10/31/16 Lecture 11 outline

• WKB (Wentzel, Kramers, Brillouin) approximation, continued. For high momentum, $\psi_E(x)$'s wiggles are smaller than $V(x)$'s wiggles, so can approximate solutions via $V(x) \approx$ constant and then add successive corrections. Write the time-indep SE in terms of $k(x) = \sqrt{2m(E-V(x))/\hbar^2}$ or $k(x) \equiv -i\sqrt{2m(V(x)-E)/\hbar^2}$ in $E < V$ and $E > V$ regions respectively, so

$$\psi_E'' + k(x)^2\psi_E(x) = 0.$$

Take $\psi_E(x) \equiv e^{iW(x)/\hbar}$ to get

$$i\hbar W'' - (W')^2 + \hbar^2 k^2 = 0.$$

So for $\hbar|W''|^2 \ll |W'|^2$ we end up with $W_0'(x) = \pm\hbar k(x)$, where the 0 means leading order. Recall that we saw (in the context of probability conservation that this was Hamilton's equation, in the classical limit. We plug this back in to get an iterative equation for W_{n+1} in terms of W_n. In particular, $(W_0 + \hbar W_1)' = \pm\sqrt{\hbar^2 k(x)^2 + i\hbar W_0''}$ where expanding the square-root (validity of the WKB approximation requires that $k' \ll k^2$) and integrating gives

$$\psi_E \approx e^{i(W_0+\hbar W_1)/\hbar)} \approx |k(x)|^{-1/2}\exp[\pm i\int^x dx' k(x')].$$

Note that $|\psi_E|^2 \approx |k(x)|^{-1} \sim 1/v(x)$, which agrees with what one might call the classical likelihood of finding a particle with velocity v in some region dx, since $dx/v = dt$ is the time that it spends in that region.

• We have to patch together these solutions across the values of x where $E = V$; in those vicinities can approximate in terms of the linear potential, with the Airy function. Suppose that there are classical turning points at $x = x_1$ and $x = x_2$, so the classical motion is for $x_1 \leq x \leq x_2$, which is called region II. Regions I and III are the classically forbidden regions $x < x_1$ and $x > x_2$. Match the WKB solution in region II to the asymptotic behavior of the Airy function at the turning point, where V is approximately linear: $Ai(z) \to z^{-1/4}(2\sqrt{\pi})^{-1}e^{-2z^{3/2}/3}$ for $z \to \infty$ and $A_i(z) \to |z|^{-1/4}\pi^{-1/2}\cos(2/3|z|^{3/2} - \pi/4)$ for $z \to -\infty$. So get

$$\psi_{E,I\to II} \to 2(E-V(x))^{-1/4}\cos\left(\hbar^{-1}\int_{x_1}^x dx'\sqrt{2m(E-V(x'))} - \pi/4\right),$$

$$\psi_{E,III\to II} \to 2(E-V(x))^{-1/4}\cos\left(-\hbar^{-1}\int_x^{x_2} dx'\sqrt{2m(E-V(x'))} + \pi/4\right),$$

and the two must agree. So the argument of the cos must differ by $n\pi$. The upshot is that, if x_1 and x_2 are two classical turning points, these approximations lead to $\int_{x_1}^{x_2} dx\sqrt{2m[E-V(x)]} = (n+\frac{1}{2})\pi\hbar$, like the Born Sommerfield Wilson quantization $\oint pdq = 2\pi n\hbar$. Note that for e.g. the SHO the classical solution is $x = A\cos(\omega t+\phi)$, $p = m\dot{x} = -m\omega A\sin(\omega t+\phi)$, $\oint pdq = \int_0^{2\pi/\omega} A^2 m\omega^2 \sin^2(\omega t+\phi)dt = \pi m\omega A^2 = 2\pi E/\omega$, so the WKB quantization rule gives $E_n = (n+\frac{1}{2})\hbar\omega$, so in this case it gives the exact result. More generally, it gives a good approximation for E_n when $n \gg 1$.

- Also, tunneling through a barrier: probability $\sim e^{-2\int_{x_1}^{x_2}\sqrt{2m(V_{eff}(x)-E)}dx/\hbar}$, where x here could also denote the radial direction of a 3d system.
- Example: semi-infinite SHO and spectrum from keeping odd parity solutions.
- Example: particle of mass m in gravity, on a $V=\infty$ floor. Can use the same trick: take $V = mg|x|$ and restrict to $x>0$ by keeping only parity odd solutions. The exact solution can be found from the Airy function's zeros. The approximate solution can be found from the WKB method. The WKB approximation for the energy levels is found from:

$$\int_{-E/mg}^{E/mg} dx\sqrt{2m(E-mg|x|)} = 2\int_0^{E/mg} dx\sqrt{2m(E-mgx)} = (n_{odd}+\tfrac{1}{2})\pi\hbar$$

Writing $n_{odd} = 2n-1$ with $n = 1,2,3\ldots$ and doing the integral gives $E_n^{WKB} = \frac{1}{2}(3(n-\frac{1}{4})\pi)^{2/3}(mg^2\hbar^2)^{1/3}$. This agrees extremely well with the result from the zeros of $Ai(z)$, even for low n, and the agreement gets better and better as n is increased.

- Propagator: $K(x_2,t_2;x_1,t_1) \equiv \langle x_2|U(t_2,t_1)|x_1\rangle$. Evaluate by inserting complete set of energy eigenstates. E.g. for free particle:

$$K_{free} = \int \frac{dp}{2\pi\hbar}\exp[i(p(x_2-x_1)-p^2(t_2-t_1)/2m)/\hbar] =$$

$$= \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\exp[im(x_2-x_1)^2/2\hbar(t_2-t_1)].$$

For the SHO get

$$K_{SHO} = \sum_n u_n(x_2)u_n^*(x_1)e^{-iE_n(t_2-t_1)/\hbar} =$$

$$\sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega(t_2-t_1))}}\exp[im\omega\left((x_2^2+x_1^2)\cos\omega(t_2-t_1)-2x_2x_1\right)/2\hbar\sin(\omega(t_2-t_1))].$$

These look a bit disgusting but are actually nice: the exponentials are the expected Hamilton functions from classical mechanics, fitting with our discussion before. The fact that they are precisely the classical result, without additional quantum corrections, is special to cases where every term in the Hamiltonian is at most quadratic. In terms of the path integral, the WKB approximation is related to a saddle point approximation of integrals, and the integrals reduce to Gaussians for the case of quadratic actions, and the saddle point approximation in such special cases happens to be exact.

E.g. for a free particle we can evaluate $S[x_{cl}, \dot{x}_{cl}] = \int_{t_1,x_1}^{t_2,x_2} dt \frac{1}{2} m\dot{x}^2 = \frac{1}{2}m(x_1 - x_2)^2/(t_1 - t_2)$. For a SHO, $S[x_{cl}, \dot{x}_{cl}] = \int dt(\frac{1}{2}A^2 m\omega^2)(\sin^2(\omega t + \phi) - \cos^2(\omega t + \phi)) = \ldots$ where we eliminate A and ϕ in terms of (x_1, t_1) and (x_2, t_2).

- Next time: path integral formulation of QM.