10/26/16 Lecture 10 outline

• Charged particles in electric and magnetic fields. Gauge invariance. Recall from classical mechanics that $L = L_0 + \frac{q}{c}\vec{v} \cdot \vec{A} - q\phi$, so $\vec{p} = \vec{p}_0 + \frac{q}{c}\vec{A}$. Aside on relativity: $\int dt(-c\phi + \vec{v} \cdot \vec{A})(q/c) = -q/c \int A_\mu dx^\mu$ is Lorentz invariant. Then $H = (\vec{p} - q\vec{A}/c)^2/2m + V(\vec{x}) + q\phi$. Replace $\vec{p} \to -i\hbar\nabla$ in QM in position space. Gauge invariance: $\vec{A} \to \vec{A} + \nabla f$, $\phi \to \phi - \partial f/c\partial t$ preserves $\vec{E} = -\nabla\phi - \partial\vec{A}/c\partial t$. In QM it affects the phase of the wavefunction $\psi \to e^{iqf/\hbar c}\psi$, but is an exact symmetry of any and all physics. Fundamental in high energy physics: forces = gauge symmetries.

Aharanov-Bohm / Dirac effect: use $\psi \sim e^{iS/\hbar}$ and compare interference on two paths, on two sides of solenoid: $\psi_1/\psi_2 = e^{i(S_1 - S_2)/\hbar}$ and note that $(S_1 - S_2) = \oint (q/c)\vec{A} \cdot d\vec{\ell} = q\Phi/c$, where Φ is the magnetic flux. So e.g. $\psi_1 = \psi_2$ if $q\Phi = 2\pi\hbar cn$, which if we set $\Phi = 4\pi q_{mag}$ is Dirac's quantization rule. Magnetic monopoles could be the explanation of electric charge quantization.

• Particle in $V(x) = V_0\theta(x)$ (step function, whose derivative is the delta function). Consider cases $E > V_0$ and $E < V_0$.

• WKB (Wentzel, Kramers, Brillouin) approximation. For high momentum, $\psi_E(x)$'s wiggles are smaller than $V(x)$'s wiggles, so can approximate solutions via $V(x) \approx$ constant and then add successive corrections. Write the time-indep SE in terms of $k(x) = \sqrt{2m(E - V(x))/\hbar^2}$ or $k(x) \equiv -i\sqrt{2m(V(x) - E)/\hbar^2}$ in $E < V$ and $E > V$ regions respectively, so

$$\psi_E'' + k(x)^2\psi_E(x) = 0.$$

Take $\psi_E(x) \equiv e^{iW(x)/\hbar}$ to get

$$i\hbar W'' - (W')^2 + \hbar^2 k^2 = 0.$$

So for $\hbar|W''|^2 \ll |W'|^2$ we end up with $W_0'(x) = \pm\hbar k(x)$. Define $W(x) = \sum_{n=0}^\infty \hbar^n W_n(x)$ and plug back in to get an iterative equation for W_{n+1} in terms of W_n. In particular, $(W_0 + \hbar W_1)' = \pm\sqrt{\hbar^2 k(x)^2 + i\hbar W_0''}$ where expanding the square-root and integrating gives

$$\psi_E \approx e^{i(W_0 + \hbar W_1)/\hbar)} \approx |k(x)|^{-1/2} \exp[\pm i \int^x dx' k(x')].$$

Note that $|\psi_E|^2 \approx |k(x)|^{-1} \sim 1/v(x)$, which agrees with what one might call the classical likelihood of finding a particle with velocity v in some region dx, since $dx/v = dt$ is the time that it spends in that region.

• We have to patch together these solutions across the values of x where $E = V$; in those vicinities can approximate in terms of the linear potential, with the Airy function. Suppose that there are classical turning points at $x = x_1$ and $x = x_2$, so the classical motion is for $x_1 \leq x \leq x_2$, which is called region II. Regions I and III are the classically forbidden regions $x < x_1$ and $x > x_2$. Match the WKB solution in region II to the asymptotic behavior of the Airy function: $Ai(z) \to z^{-1/4}(2\sqrt{\pi})^{-1}e^{-2z^{3/2}/3}$ for $z \to \infty$ and $A_i(z) \to |z|^{-1/4}\pi^{-1/2}\cos(2/3|z|^{3/2} - \pi/4)$ for $z \to -\infty$. So get

$$\psi_{E,I\to II} \to 2(E - V(x))^{-1/4}\cos\left(\hbar^{-1}\int_{x_1}^{x} dx'\sqrt{2m(E - V(x'))} - \pi/4\right),$$

$$\psi_{E,III\to II} \to 2(E - V(x))^{-1/4}\cos\left(-\hbar^{-1}\int_{x}^{x_2} dx'\sqrt{2m(E - V(x'))} + \pi/4\right),$$

and the two must agree. So the argument of the cos must differ by $n\pi$. The upshot is that, if x_1 and x_2 are two classical turning points, these approximations lead to $\int_{x_1}^{x_2} dx\sqrt{2m[E - V(x)]} = (n + \frac{1}{2})\pi\hbar$, like the Born Sommerfield Wilson quantization $\oint pdq = 2\pi n\hbar$. Note that for e.g. the SHO the classical solution is $x = A\cos(\omega t + \phi)$, $p = m\dot{x} = -m\omega A\sin(\omega t + \phi)$, $\oint pdq = \int_0^{2\pi/\omega} A^2 m\omega^2 \sin^2(\omega t + \phi)dt = \pi m\omega A^2 = 2\pi E/\omega$, so the WKB quantization rule gives $E_n = (n + \frac{1}{2})\hbar\omega$, so in this case it gives the exact result. More generally, it gives a good approximation for E_n when $n \gg 1$.