212 Homework 2, due 10/21/16

1. Let $\langle x|\psi\rangle \equiv \psi(x) = e^{ikx}\sqrt{P(x)}$, where $P(x) \equiv (\sigma\sqrt{2\pi})^{-1}e^{-(x-x_0)^2/2\sigma^2}$ is a standard, properly normalized, Gaussian distribution with standard deviation σ and average x_0.
   (a) Compute $\langle x\rangle$ and $\langle x^2\rangle$ in the state $\psi(x)$. Find $\langle(\Delta x)^2\rangle$.
   (b) Compute $\tilde{\psi}(p) \equiv \langle p|\psi\rangle$ and show that $\tilde{P}(p) \equiv |\tilde{\psi}(p)|^2$ (note: $\tilde{P}(p)$ is not the Fourier transform of $P(x)$)is also a Gaussian. Find its average p_0 and standard deviation $\tilde{\sigma}$.
   (c) Compute $\langle p\rangle$ and $\langle p^2\rangle$ in both the x basis and the p basis. Verify that they agree.
   (d) Verify that the Gaussian saturates the inequality $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{1}{4}|\langle[x,p]\rangle|^2$.

2. Consider the functions
$$\psi_{n=1,2,\ldots}(x) \equiv \langle x|n\rangle \equiv \begin{cases} \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L}) & \text{for } 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Verify that $\int_{-\infty}^{\infty}\psi_m(x)^*\psi_n(x) \equiv \langle m|n\rangle = \delta_{n,m}$.
   (b) Compute $\langle x\rangle$ and $\langle x^2\rangle$ in $|\psi\rangle = |n\rangle$ for general n.
   (c) Compute $\langle p\rangle$ and $\langle p^2\rangle$ in $|\psi\rangle = |n\rangle$ for general n.
   (d) Verify that the uncertainty principle is satisfied for general $|n\rangle$ states.

3. Using $\vec{L} = \vec{r}\times\vec{p}$ and the $\vec{x}$ and $\vec{p}$ commutation relations, verify that $[L_a, L_b] = \sum_{c=1}^{3} i\hbar\epsilon_{abc}L_c$.

4. Sakurai 2.1: Take $H = \omega S_z$ for the two-state system. Write the Heisenberg equations of motion for $S_x(t)$, $S_y(t)$, and $S_z(t)$. Solve them to find these operators as a function of time.

5. Sakurai 2.3: Again, $H = \omega S_z$. At time $t = 0$ the electron is in an eigenstate of $\widehat{n}\cdot\vec{S}$ (similar to HW 1). Take $\widehat{n}$ to be a unit vector with $\phi = 0$ (i.e. in the xz plane, with angle θ from the z axis).
   (a) Find the probability to measure $S_x = +\hbar/2$ as a function of t.
   (b) Find $\langle S_x\rangle$ as a function of time.
   (c) Answer the above for general θ, and verify that the answers make sense for the cases $\theta = 0$ and $\theta = \pi/2$.

6. Sakurai 2.12: 1d SHO with $|\psi(t=0)\rangle = \exp(-i\widehat{p}L/\hbar)|0\rangle$. Evaluate $\langle x(t)\rangle$ for all $t > 0$ in this state, using the Heisenberg picture.

7. Sakurai 2.14: 1d SHO.

(a) Using the creation and annihilation operators, evaluate the matrix elements of the operators x, p, $(xp+px)$, x^2, and p^2, all between $\langle m|$ and $|n\rangle$. Evaluate them as functions of t.

(b) Verify that the viral theorem holds for the expectation values of KE and PE, taken with respect to an energy eigenstate.

8. Sakurai 2.17: Consider the 1d SHO.

(a) Find a linear combination of $|0\rangle$ and $|1\rangle$ that maximizes $\langle x\rangle$.

(b) Suppose that $|\psi(t=0)\rangle$ is the state from part (a). Find $\langle x\rangle$ for all $t>0$, using both the Schrodinger and Heisenberg pictures.

(c) Evaluate $\langle(\Delta x)^2\rangle$ for all $t>0$ in either picture.

9. Sakurai 2.19. SHO coherent states:

(a) Verify that $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ is a normalized coherent state, i.e. that $\langle\lambda|\lambda\rangle = 1$ and $a|\lambda\rangle = \lambda|\lambda\rangle$.

(b) Compute $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ in the coherent state, and verify that the uncertainty principle is satisfied or saturated.

(c) Write $|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle$, show that $|f(n)|^2$ is a Poisson distribution in n. Find the most probable value of n.

(d) Show that the coherent state can also be found by applying the translation operator to the ground state.