212 Homework 1, due 10/7/16

1. (Sakurai 1.9.) Construct $|+_{\widehat{n}}\rangle \equiv |\vec{S}\cdot\widehat{n};+\rangle$ by explicitly solving the eigenvalue equation, letting $\widehat{n}$ be a unit vector with the usual angles (θ,ϕ) of spherical coordinates.

2. Suppose a SG setup measures $\vec{S}\cdot\widehat{n} = \hbar/2$. What is the probability that a subsequent SG setup will measure $\vec{S}\cdot\widehat{n}' = -\hbar/2$? Take $\widehat{n}$ to have angles (θ,ϕ) and $\widehat{n}'$ to have angles (θ',ϕ'). Verify that you get reasonable answers if $\widehat{n} = \widehat{n}'$ and $\widehat{n} = -\widehat{n}'$.

3. Evaluate $\langle S_z\rangle$ and $\langle S_z^2\rangle$ in the state $|+_{\widehat{n}}\rangle$.

4. A beam of spin 1/2 atoms go through a series of SG measurements as follows.

   (a) The first measurement is along the $\widehat{z}$ axis and blocks out spin down (only spin up goes out).

   (b) The second is along the $\widehat{n}$ axis and blocks spin down.

   (c) The second is along the $\widehat{z}$ axis and blocks spin up.

   Find the probability to make it through all three, in terms of the (θ,ϕ) of $\widehat{n}$.

5. Let σ^i be the Pauli matrices.

   (a) Verify that $\vec{a}\cdot\vec{\sigma}\vec{b}\cdot\vec{\sigma} = i(\vec{a}\times\vec{b})\cdot\vec{\sigma} + \mathbf{1}\vec{a}\cdot\vec{b}$ for arbitrary vectors $\vec{a}$ and $\vec{b}$. Equivalently you can show $\sigma^i\sigma^j = i\epsilon_{ijk}\sigma^k + \delta_{ij}\mathbf{1}$.

   (b) Show by series expansion that $e^{i\theta\widehat{n}\cdot\vec{\sigma}} = \cos\theta\mathbf{1} + i\sin\theta\widehat{n}\cdot\vec{\sigma}$.

   (c) Verify that $e^{i\theta\widehat{n}\cdot\vec{\sigma}}$ is unitary, provided that θ is real.

6. Let $A = a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma}$ and $B = b_0\mathbf{1} + \vec{b}\cdot\vec{\sigma}$.

   (a) What is the condition on the a_0, $\vec{a}$, b_0, $\vec{b}$ for A and B to be Hermitian?

   (b) Compute $[A,B]$. What condition is needed for $[A,B]=0$?

   (c) Compute $\{A,B\}$.

7. Suppose that $A = a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma}$ corresponds to a physical observable.

   (a) If the A observable is measured, what are the possible outcomes?

   (b) Suppose that an electron goes through a SG setup that measures $S_z = \hbar/2$. A subsequent experiment measures the A observable. What is the probability for each outcome? What is $\langle A\rangle$?

   (c) What condition is needed (for a_0 and a^i) such that A and S_z measurements don't interfere with each other?

8. Show that the projector onto $|+_{\widehat{n}}\rangle$ can be written in the usual $|\pm_{\widehat{z}}\rangle$ basis as $P = |+_{\widehat{n}}\rangle\langle+_{\widehat{n}}| \to \frac{1}{2}(1+\widehat{n}\cdot\vec{\sigma})$. Verify that $P^2=P$, as required for a projection operator.

9. (a) Read or convince yourself that $\langle\psi|\psi\rangle \geq 0$ implies (as discussed in class) $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$ which then implies (as started to discuss at the end of the last lecture) $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$. You don't need to write the arguments out - just convince yourself that you known them.

(b) Verify that $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$ is satisfied in the state $|+_z\rangle$ for $A = S_x$, $B = S_y$ and also for $A = S_x$, $B = S_z$.