10/26 Lecture outline

⋆ **Reading for today's lecture: Coleman lecture notes pages 103-120 (skip parts about counterterms for now).**

• Last time:

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x :\mathcal{H}_I:(x)}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

computing in the toy model for nucleons and mesons:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

So $\mathcal{H}_I = g\phi\psi^\dagger\psi$. Recall $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

We considered $N + N \to N + N$, to $\mathcal{O}(g^2)$. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1 - p_1')^2 - \mu^2} + \frac{1}{(p_1 - p_2')^2 - \mu^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2+m^2}, p\widehat{e})$ and $p_2 = (\sqrt{p^2+m^2}, -p\widehat{e})$, $p_1' = (\sqrt{p^2+m^2}, p\widehat{e}')$, $p_2' = (\sqrt{p^2+m^2}, -p\widehat{e}')$, where $\widehat{e}\cdot\widehat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1-\cos\theta) + \mu^2} + \frac{1}{2p^2(1+\cos\theta) + \mu^2}\right).$$

According to the above, $[\mathcal{A}(2\to 2)] = 0$ and the above is consistent with that. Good.

Note also that the amplitude is symmetric if we exchange $p_1^\mu \leftrightarrow p_2^\mu$ and likewise for the outgoing states. This fits with the fact that the N states are identical bosons, which follows from the fact that $[\psi(t,\vec{x}), \psi(t,\vec{y})] = 0$. As we'll discuss later, identical fermions instead have $\{\psi(t,\vec{x}), \psi(t,\vec{y})\} = 0$.

• Recall how we got the above answer. We expand $\exp(-ig\int d^4x\mathcal{H})$ and compute the time ordered expectation values using Wick's theorems, with the contractions giving factors of $D_F(x_1 - x_2)$. Doing this, we get a $\int d^4x$ for each factor of $-ig$ and a d^4k for each internal contraction. Draw a picture in position space. Let E be the number of external lines, i.e. the number of incoming + outgoing particles. (We saw last time that $[\mathcal{A}] = 4 - E$.) It is easier to think about everything in momentum space. Then the $\int d^4x$ for each vertex gives a $(2\pi)^4\delta^4(p_{total,\ in})$.

• Feynman rules! Each vertex gets $(-ig)(2\pi)^4\delta^4(p_{total\ in})$, each internal line gets $\int \frac{d^4k}{(2\pi)^4} D_F(k^2)$, where D_F is the propagator, e.g. $D_F(k^2) = \frac{i}{k^2-m^2+i\epsilon}$. Result is $\langle f|(S-1)|i\rangle$, so divide by $(2\pi)^4\delta^4(p_F - p_I)$ to get $i\mathcal{A}_{fi}$.

If the diagram has no loops, the momentum conserving delta functions fix all internal momenta in terms of the external ones. When the diagram has $L \neq 0$ loops, the procedure above yields integrals over the internal momenta of the loops. (Note that if a diagram has I internal lines and V vertices, then there are I momentum integrals, and V momentum conserving delta functions; one of these becomes overall momentum conservation, so there are $L = I - V - 1$ momentum integrals left to do, and L is the number of loops in the diagram.) Any loop momentum integrals require renormalization, which we'll discuss later (next quarter), so for now we'll just consider "tree-level" contributions, associated with diagrams without loops, $L = 0$.

• More examples:

(1) $N(p_1) + \bar{N}(p_2) \to N(p_1') + \bar{N}(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - \mu^2} + \frac{i}{(p_1 + p_2) - \mu^2} \right).$$

(2) $N(p_1) + \bar{N}(p_2) \to \phi(p_1')\phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - m^2} + \frac{i}{(p_1 - p_2') - m^2} \right).$$

(3) $N(p_1) + \phi(p_2) \to N(p_1') + \phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_2') - m^2} + \frac{i}{(p_1 + p_2) - m^2} \right).$$

Note: the $1/2!$ from expanding $e^{-i\int d^4x \mathcal{H}_I(x)}$ is cancelled by a factor of 2 from exchanging the two vertices.