10/19 Lecture outline

⋆ **Reading for today's lecture: Coleman lecture notes pages 70-80.**

• Last time:

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x :\mathcal{H}_I:(x)}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

The initial states have momenta $p_1 \ldots p_n$ and the final states have momenta $q_1 \ldots q_m$. Need to strip off the momentum conserving delta function to get the amplitude. Note the normal ordering in $:\mathcal{H}_I:(x)$: we want the full Hamiltonian to be normal ordered so e.g. $|0\rangle$ has zero energy. If we don't normal order here, we'll have issues with some loops for the interaction. Later (next quarter) we'll discuss counter terms, and essentially here the normal ordering is throwing away things that we'd anyway have to subtract off with counterterms.

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the toy model for nucleons and mesons:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

So $\mathcal{H}_I = g\phi\psi^\dagger\psi$.

Note that there is a $\psi \to e^{i\alpha}\psi$ global symmetry, so there is a corresponding conserved current and charge, which we'll call "nucleon number". We choose to assign ψ nucleon number charge -1 and $\psi^\dagger$ nucleon number $+1$. (This is similar to the recent HW question.)

• Time out for a few comments. The scalar ϕ and coupling $g\phi\psi\bar{\psi}$ is also a good toy model for the Higgs' coupling to fermions. Here, and there, the scalar "Yukawa" coupling mediates a force. The strong, weak, and electromagnetic forces are communicated by spin 1 gauge fields. Gravity is mediated by the spin 2 graviton (and the difference between spin 1 vs spin 2 is part of why quantum gravity is conceptually and technically challenging). Spin 0 scalars can also mediate forces, as in this example. We'll see that their force is always attractive (even spins always lead to attractive forces). Fifth force experimental bounds strongly constrain the existence, mass, and couplings of fundamental scalars.

In our toy model, where ψ and $\bar{\psi}$ are scalars, the theory has a vacuum instability, since a cubic potential isn't bounded below. This shows up only indirectly in perturbation theory, and is more of a non-perturbative issue.

• Use $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Examples of states:

$$|\phi(p)\rangle = a^\dagger(p)|0\rangle, \qquad |N(p)\rangle = b^\dagger(p)|0\rangle, \qquad |\bar{N}(p)\rangle = c^\dagger(p)|0\rangle.$$

Note then e.g.

$$\langle 0|\phi(x)|\phi(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi(x)|N(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi^\dagger(x)|N(p)\rangle = 0.$$

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$, i.e. $\mathcal{A} = -g$. Probability $\sim g^2$.

Let's also do some dimensional analysis. Recall that $[\phi] = 1$ and $[d^3k/2\omega] = 2$, so $[a(k)] = [a^\dagger(k)] = -1$. So $[|i, f\rangle] = -n_{i,f}$ and $[\mathcal{A}] = 4 - n_i - n_f$. In our example, $[g] = 1$, so $\mathcal{A}(\phi \to N + \bar{N}) = -g$ is dimensionally consistent. Good.

Comment: draw pictures to illustrate a $\sim g^3$ correction, with 1 loop. In general, amplitudes scale like $(g^2/16\pi^2)^L$ where L is the number of loops. But we'll see that loops lead to divergent momenta integrals, eg. $\int^\Lambda d^4k/k^2 - m^2 \sim \Lambda^2$. How to handle this will be deferred to next quarter...

• Now consider $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is **(don't forget the time ordering!)**

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S - 1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

Continue next time...