10/14 Lecture outline

⋆ **Reading for today's lecture: Coleman lecture notes pages 70-80.**

• Last time: contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator. Let e.g. $\phi(x) = \phi^+(x) + \phi^-(x)$, where ϕ^+ is the term with annihilation operators and ϕ^- is the one with creation operators (using Heisenberg and Pauli's reversed-looking notation). Then for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.

Wick's theorem (we'll soon see it's useful, since S-matrix elements will involve T ordered correlation functions):

$$T(\phi_1 \dots \phi_n) =: \phi_1 \dots \phi_n : + \sum_{contractions} : \phi_1 \dots \phi_n :$$
$$=: e^{\frac{1}{2}\sum_{i,j=1}^{n} C(\phi_i \phi_j)\frac{\partial}{\partial \phi_i}\frac{\partial}{\partial \phi_j}} \phi_1 \dots \phi_n :$$

(where C is the contraction symbol) to get rid of the time ordered products.

Prove Wick's theorem by iteration: define the RHS as $W(\phi_1 \dots \phi_n)$ and we assume $T(\phi_2 \dots \phi_n) = W(\phi_2 \dots \phi_n)$ and want to prove then that $T(\phi_1 \dots \phi_n) = W(\phi_1 \dots \phi_n)$. WLOG, take $t_1 > t_2 \dots t_n$ so $T(\phi_1 \dots \phi_n) = \phi_1 T(\phi_2 \dots \phi_n) = \phi_1 W(\phi_2 \dots \phi_n) = \phi_1^- W + W\phi_1^+ + [\phi_1^+, W]$. The first two terms are normal ordered and give all contractions not involving ϕ_1, while the last gives all normal ordered contractions involving ϕ_1.

So note that
$$\langle T(\phi_1 \dots \phi_n)\rangle \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\text{fullycontracted}} & \text{for } n \text{ even.} \end{cases}$$

• Simple examples of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) - \rho(x)\phi$$

with $\rho(x)$ an external source forcing function. We'll show this theory is exactly solvable and gives probability for particle creation given by the Poisson distribution.

Next example:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• In QM we can use the S-picture, $i\hbar\frac{d}{dt}|\psi(t)\rangle_S = H|\psi\rangle_S$, or the H-picture, where t is in the operators $i\hbar\frac{d}{dt}\mathcal{O}_{\mathcal{H}}(t) = [\mathcal{O}_H, H]$.

In interacting theories, it is useful to use the hybrid, interaction picture. Write $H = H_0 + H_{int}$. We use H_0 to time evolve the operators, and H_{int} to time evolve the states:

$$i\frac{d}{dt}\mathcal{O}_I(t) = [\mathcal{O}_I, H_0], \qquad i\frac{d}{dt}|\psi(t)\rangle_I = H_{int}|\psi(t)\rangle_I.$$

$$|\psi(t)\rangle_I = e^{iH_0(q_S,p_S)t}|\psi(t)\rangle_S, \qquad \mathcal{O}_I = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}$$

For example, we'll take H_0 to be the free Hamilton of KG fields, with only the mass terms included in the potential. Again, this is free because the EOM are linear, and we can solve for $\phi(x)$ by superposition. $H_I(t)$ is built from these free fields

$$\phi(\vec{x}, t) = e^{iH_0t}\phi_S(\vec{x})e^{-iH_0t}.$$

As before, upon quantization, the fields become superpositions of creation and annihilation operators. The states are all the various multiparticle states, coming from acting with the creation operators on the vacuum. Time evolution is via the interaction picture operator that satisfies

$$i\frac{d}{dt}U_I(t,t') = H_I(t)U_I(t,t').$$

• Compute probabilities from squaring amplitudes, and amplitudes from $\langle f(t = +\infty)|i(t = -\infty)\rangle = \langle f|S|i\rangle = \langle f|U(\infty, -\infty)|i\rangle$. Naively, $U(t_f, t_i) = \exp(-\frac{i}{\hbar}\int_{t_i}^{t_f} H_{int}(t)dt)$, but have to be careful about H_{int} not commuting at different times. Get time ordering.

• Dirac's / Dyson's formula:

$$U_I(t,t') = Te^{-i\int_{t'}^{t} dt''}H_I(t'').$$

Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle (see Coleman notes for more details).

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Derive it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does. Illustrate it for 2nd term $(-i)^2/2! \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T(H_I(t_1)H_I(t_2)$, get twice the integral over the $t_1 > t_2$ region instead of the integral over the square.

• Now use Wick's theorem to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x\mathcal{H}_I(x)}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

The initial states have momenta $p_1 \dots p_n$ and the final states have momenta $q_1 \dots q_m$. Need to strip off the momentum conserving delta function to get the amplitude.

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3xg\phi\psi^\dagger\psi$. Use $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Examples of states:

$$|\phi(p)\rangle = a^\dagger(p)|0\rangle, \qquad |N(p)\rangle = b^\dagger(p)|0\rangle, \qquad |\bar{N}(p)\rangle = c^\dagger(p)|0\rangle.$$

Note then e.g.

$$\langle 0|\phi(x)|\phi(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi(x)|N(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi^\dagger(x)|N(p)\rangle = 0.$$

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$, i.e. $\mathcal{A} = -g$. Probability $\sim g^2$.

Comment: draw pictures to illustrate a $\sim g^3$ correction, with 1 loop. In general, amplitudes scale like $(g^2/16\pi^2)^L$ where L is the number of loops. But we'll see that loops lead to divergent momenta integrals, eg. $\int^\Lambda d^4k/k^2 - m^2 \sim \Lambda^2$. How to handle this will be deferred to next quarter...