12/2 Lecture outline

⋆ **Reading: Tong chapter 6**

• Last time: For the massive vector mesons, write down the general lagrangian:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A^\nu \partial_\mu A^\nu + a\partial_\mu A^\mu \partial_\nu A^\nu + bA_\mu A^\mu).$$

The sign is chosen so that the kinetic terms of the spatial components have the right sign. Write the EOM:

$$-\partial^2 A_\nu - a\partial_\nu(\partial \cdot A) + bA_\nu = 0,$$

and note plane wave solutions $A_\mu(x) = \epsilon_\nu e^{-ik\cdot x}$ solves it if $k^2\epsilon_\nu + ak_\nu(k\cdot\epsilon) + b\epsilon_\nu = 0$. The longitudinal solutions have $\epsilon \propto k$ and have mass $\mu_L^2 = -b/(1+a)$. The transverse have mass $\mu_T^2 = -b$. Can eliminate the uninteresting longitudinal solution by taking $a = -1$ and $b \neq 0$, then write Proca lagrangian in terms of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mu^2 A_\mu A^\mu$$

Each component A_μ satisfies the KG equation with mass μ. Can choose $\epsilon^{(\pm)} = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$ and $\epsilon^{(0)} = (0, 0, 0, 1)$, where the label is the value of J_z of the spin 1 vector. Normalize by $\epsilon^{(r)*}\cdot\epsilon^{(s)} = -\delta^{rs}$ and $\sum_r \epsilon_\mu^{(r)*}\epsilon_\nu^{(r)} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$.

The conjugate momenta to A_μ are $\pi^0 = \partial\mathcal{L}/\partial\dot{A}_0 = 0$, and $\pi^i = \partial\mathcal{L}/\partial\dot{A}_i = -F^{0i} = E^i$. Then $\mathcal{H} = -\frac{1}{2}(F_{0i}F^{0i} - \frac{1}{2}F_{ij}F^{ij} + \mu^2 A_i A^i - \frac{1}{2}\mu^2 A_0 A^0) \geq 0$.

• Quantize the massive vector:

$$[A_i(t,\vec{x}), F^{j0}(t,\vec{y})] = i\delta_i^j \delta^{(3)}(\vec{x}-\vec{y}).$$

In terms of the plane wave solutions,

$$A_\mu(x) = \sum_{r=1}^{3}\int \frac{d^3k}{(2\pi)^3(2\omega_k)}\left[a_k^r \epsilon_\mu^r e^{-ikx} + a_k^{\dagger r}\epsilon_\mu^{*r} e^{ikx}\right],$$

(as usual, there is a choice of convention in the normalization of the creation and annihilation operators), and with this normalization the quantization condition implies that

$$[a_k^r, a_{k'}^{\dagger s}] = \delta^{rs}(2\pi)^3(2\omega_k)\delta^3(\vec{k}-\vec{k}').$$

and

$$:\mathcal{H} := \sum_r \int \frac{d^3k}{(2\pi)^3(2\omega_k)}\omega_k a_k^{\dagger r} a_k^r.$$

The propagator, the contraction of $A_\mu(x)$ and $A_\nu(y)$, is

$$\langle TA_\mu(x)A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\left[\frac{-i(g_{\mu\nu}-k_\mu k_\nu/\mu^2)}{k^2-\mu^2+i\epsilon}\right].$$

So the Feynman rule is that massive vectors have the momentum space propagator

$$\left[\frac{-i(g_{\mu\nu}-k_\mu k_\nu/\mu^2)}{k^2-\mu^2+i\epsilon}\right].$$

And $\langle 0|A_\mu(x)|V(k,r)\rangle = \epsilon_\mu(k)^r e^{-ikx}$, so incoming vector mesons have $\epsilon_\mu^r(k)$ and outgoing have $\epsilon^{*r}(k)$.

We can couple the massive vector to other fields, e.g. to a fermion via $\mathcal{L}_{int} = -g\bar{\psi}\not{A}\Gamma\psi$, with $\Gamma = 1$ (vector) or $\Gamma = \gamma_5$ (axial vector). Gives Feynman rule that a vertex has a factor of $-ig\gamma^\mu\Gamma$.

• Now consider the massless theory. If we add $\mathcal{L} \supset -A_\mu j^\mu$ to the massive theory, get $\partial_\mu A^\mu = \mu^{-2}\partial_\mu j^\mu$, so there is only a sensible limit if $\partial_\mu j^\mu = 0$, must couple to a conserved current. Associate with symmetry, $\psi \to e^{-i\lambda q}\psi$, where q is the charge. The massless theory must be associated with gauge invariance: can make above symmetry transformations where $\lambda = \lambda(x)$ is a local function, and this is a redundancy, rather than a symmetry, when combined with $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\lambda(x)$, where e is a coupling constant. Consider minimal coupling: replace $\partial^\mu \to D^\mu = \partial^\mu + ieA^\mu q$ for a charge q field to ensure that the theory respects gauged version of the symmetry.

Another way to say it: the only way to have a sensible $\mu \to 0$ limit is if A_μ is a gauge field, associated with a local gauge symmetry. The reason is that the operator in brackets in

$$[\eta_{\mu\nu}(\partial^\rho\partial_\rho) - \partial_\mu\partial_\nu]A^\nu = 0$$

is not invertable: it annihilates any function of form $\partial_\mu\lambda$. Solution: require that $A_\mu \sim A_\mu + \partial_\mu\lambda$, i.e. gauge invariance. The space of gauge fields has equivalent gauge orbits.

Minimal coupling examples:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - eq\not{A} - m)\psi.$$

$$\mathcal{L} = D_\mu\phi^* D^\mu\phi - m^2|\phi|^2.$$

The first gives a $\bar{\psi}A_\mu\psi$ Feynman vertex weighted by $-ieq\gamma^\mu$, and the second gives a $\phi^*(p')A_\mu\phi(p)$ vertex weighted by $ieq(p+p')^\mu$, along with a $A_\mu A_\nu\phi^*\phi$ seagull graph weighted by $2ie^2q^2g^{\mu\nu}$ (factor of 2 because of the two identical A_μ fields).

As in the massive vector case, A_0 has no kinetic term, can solve its EOM ($\nabla\cdot\vec{E}=0\to\nabla^2A_0+\nabla\cdot\dot{\vec{A}}=0$):

$$A_0(\vec{x})=\int d^3\vec{x}'\frac{\nabla\cdot\dot{\vec{A}}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}.$$

Gauge fixing: can always choose e.g. $\partial_\mu A^\mu=0$. Doesn't entirely fix the gauge. Can still pick $\nabla\cdot\vec{A}=0$ – Coulomb gauge – then $A_0=0$. See two polarizations. So take $\vec{\epsilon}^r$ with $\vec{\epsilon}_r\cdot\vec{p}=0$, orthonormal. The completeness relation is similar to that above, except that we replace $\mu^2\to|\vec{p}|^2$. The propagator is then

$$\langle TA_i(x)A_j(y)\rangle=\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\left[\frac{i(\delta_{ij}-k_ik_j/|\vec{k}|^2)}{k^2+i\epsilon}\right].$$

This gauge can be a pain in the interacting theory (need to write instantaneous $\delta(x^0-y^0)/|\vec{x}-\vec{y}|$ Coulomb interaction). It's nicer to write something more manifestly Lorentz invariant.

In the massive vector case, we had the propagator $-i(g_{\mu\nu}-k_\mu k_\nu/|\mu|^2)/(k^2-\mu^2+i\epsilon)$. In the $\mu\to 0$ massless gauge theory, gauge invariance ensures that the $k_\mu k_\nu$ term has no effect in physical, on-shell amplitudes. For example, $e^+e^-\to\mu^+\mu^-$ tree-level amplitude, show that the $k_\mu k_\nu$ term in the propagator doesn't contribute for on-shell external states. Another example: Compton scattering of vector off an electron: $i\mathcal{A}=\mathcal{M}^{\mu\nu}\epsilon_\mu^{(r')*}(k')\epsilon_\nu^{(r)}(k)$. Observe that $k^\mu\mathcal{M}_{\mu\nu}=0$, decouples the helicity 0 mode. Also, square amplitude and average over initial polarizations and sum over the final ones, and note that $k^\mu\mathcal{M}_{\mu\nu}$, and likewise for k', ensures that the $1/\mu^2$ terms in the polarization completeness relation go away.

- Gauge fixing. Try to preserve Lorentz invariance by imposing $\partial_\mu A^\mu=0$, and not $A_0=0$. Can modify $\mathcal{L}$ to get Lorentz gauge EOM. More generally, can consider

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\alpha}(\partial\cdot A)^2,$$

and quantize for any parameter α. Popular choices are $\alpha=1$ (Feynman) and $\alpha=0$ (Landau). Now get $\pi^0=\partial\mathcal{L}/\partial(\dot{A}_0)=-\partial_\mu A^\mu/\alpha$. Do canonical quantization for all components, $[A_\mu(\vec{x}),\pi_\nu(\vec{y})]=i\eta_{\mu\nu}\delta(\vec{x}-\vec{y})$. Write plane wave expansion with 4 polarizations, normalized to $\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}$. Get that timelike polarizations create negative norm states. Can fix this by imposing $\partial^\mu A_\mu^+|\Psi\rangle = 0$ on the physical states, along with gauge invariance relation, to get a physical Hilbert space with positive norms.

Propagator for gauge field is

$$\langle T A_\mu(x) A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{-i(g_{\mu\nu} + (\alpha-1)k_\mu k_\nu/k^2)}{k^2+i\epsilon}\right].$$

Again, the $k_\mu k_\nu$ piece will drop out in the end in physical amplitudes. Just need to make a choice and stick with it consistently. Or keep α as a parameter, and then it's a good check on the calculation that the α indeed drops out in the end.

• QED examples:

Compton scattering, $e^-\gamma \to e^-\gamma$ (related to $e^+e^- \to \gamma\gamma$ by crossing symmetry):

$$i\mathcal{A} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}^{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + (k\to -k').\right]u^r(p)$$

$$= -i\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}^{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + (k\to -k')\right]u^r(p).$$

To compute the differential cross section, we square this and multiply it by the $2\to 2$ phase space factor. It simplifies to sum over final state spins and average over initial state ones. You can find this worked out in great detail, for the case of Compton scattering, in section 5.5 of Peskin and Schroeder.

• Ward identity: if the polarization $\epsilon^\mu(k)$ of any external photon is replaced with its 4-momentum, $\epsilon^\mu(k) \to k^\mu$, the amplitude vanishes. This can be proved in general, and it ensures that amplitudes respect gauge invariance. See Peskin and Schroeder for more details, and you'll see more about it next quarters.

Other examples, $e^+e^- \to e^+e^-$ and $e^-e^- \to e^-e^-$. The two are related by crossing symmetry. Mention $e^-e^\mp \to e^-e^\mp$ and the Coulomb potential: opposites attract and same sign charges repel. Contrast this with the scalar Yukawa case, where the potential is always attractive. Because here $\bar{v}\gamma^0 v \to +2m$, whereas in the scalar case got $\bar{v}v \to= -2m$.