11/30 Lecture outline

⋆ **Reading: Tong chapters 5, 6**

• Last time:

$$\mathcal{L} \supset \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \to \qquad \text{fermion propagator:} \qquad \frac{i}{p\!\!\!/ - m + i\epsilon},$$

$$\mathcal{L} \supset \tfrac{1}{2}\partial\phi\partial\phi - \tfrac{1}{2}\mu^2\phi^2 \qquad \to \qquad \text{scalar propagator:} \qquad \frac{i}{p^2 - \mu^2 + i\epsilon},$$

$$\mathcal{L} \supset -g\phi\bar{\psi}_a\Gamma_{ab}\psi_b(x) \qquad \to \qquad \text{scalar, fermion vertex} \qquad -ig\Gamma,$$

where the index a, b runs over the four fermion components (spin up and down for fermion and anti-fermion), so Γ is a 4×4 matrix (natural choices are $\Gamma = 1_{4\times 4}$ or $\Gamma = i\gamma_5$, where recall $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and the i is there to keep $\mathcal{L}^\dagger = \mathcal{L}$, since $(\gamma^0\gamma_5)$ is anti-hermitian).

Incoming fermions get a factor of $u^r(p)$, outgoing fermions get $\bar{u}^r(p)$; incoming antifermions gets $\bar{v}^r(p)$, and outgoing antifermions get $v^r(p)$. The amplitude has indices $r = 1, 2$ for each external fermion, which accounts for the external fermion's spin. For internal fermion propagators we sum over the four fermion indices, which is accomplished by matrix multiplication of the above tinkertoy pieces, with Tr put in as appropriate. Write the amplitude by following the arrows backwards, from the head to the tail.

$N + N \to N + N$:

$$i\mathcal{A} = -ig^2\left(\frac{\bar{u}^{s'}_{q'}\Gamma u^s_{p'}\bar{u}^{r'}_{p'}\Gamma u^r_p}{(q-q')^2 - \mu^2 + i\epsilon} - \frac{\bar{u}^{s'}_{q'}\Gamma u^r_p\bar{u}^{r'}_{p'}\Gamma u^s_q}{(q-p')^2 - \mu^2 + i\epsilon}\right).$$

• Attractive Yukawa potential for both $\psi\psi \to \psi\psi$, and also $\psi\bar{\psi} \to \psi\bar{\psi}$. Recall $\mathcal{A}_{NR} = -i\int d^3\vec{r}e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}}U(\vec{r})$. For $\psi\psi \to \psi\psi$, $\mathcal{A}_{NR} \supset -i(-ig)^2(2m)\frac{1}{(\vec{p}-\vec{p}')^2+\mu^2}$ when the spins are unchanged. Gives $U(\vec{r}) = -g^2e^{-\mu r}/4\pi r$. For $\psi\bar{\psi} \to \psi\bar{\psi}$, amplitude differs by sign, but so does $\bar{v}v$, so again get attractive potential.

Recall:

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

• Example $\Gamma = i\gamma_5$, $N + \phi \to N + \phi$, simplify $i\mathcal{A}$. Compute $|\mathcal{A}|^2$ and average over initial spins and sum over final spins. Simplify.

$$i\mathcal{A} = ig^2\bar{u}^{r'}_{p'}\gamma_5\left(\frac{p\!\!\!/ + q\!\!\!/ + m}{(p+q)^2 - m^2 + i\epsilon} + \frac{p\!\!\!/ - q\!\!\!/' + m}{(p-q')^2 - m^2 + i\epsilon}\right)\gamma_5 u^r_p,$$

$$i\mathcal{A} = ig^2\bar{u}^{(r')}(p')\not{q}u^{(r)}(p)F, \qquad F \equiv \left[\frac{1}{2p\cdot q+\mu^2+i\epsilon}+\frac{1}{2p'\cdot q+\mu^2+i\epsilon}\right].$$

$$|\mathcal{A}|^2 = g^4F^2q_\mu q_\nu Tr[\bar{u}(p')^{r'}\gamma^\mu u(p)^r\bar{u}(p)^r\gamma^\nu u(p)^{r'}].$$

Average over initial spins and sum over final ones (often physically relevant, and it simplifies the expression, using the completeness relations)

$$\tfrac{1}{2}\sum_{r,r'}|\mathcal{A}|^2 = \tfrac{1}{2}g^2F^2q_\mu q_\nu Tr[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu]$$
$$= 2g^4F^2[2(p'\cdot q)(p\cdot q) - p\cdot p'\mu^2 + m^2\mu^2].$$

• Recap: we have discussed spin 0 and spin 1/2 quantum fields. Now move up to spin 1. (Next quarter, we'll discuss renormalizability, and note there the complications with quantizing fields of spin greater than 1.) Examples with spin 1 include non-fundamental (composite) fields, e.g. spin 1 mesons, and also the fundamental force carriers: the photon, gluons, and $W^\pm$ and Z^0. The gluons and $W^\pm$ are associated with non-Abelian groups, which we'll discuss next quarter.

• Consider a spin 1 quantum field (the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group), and call it A_μ. The components of A_μ will satisfy something like a KG equation, being massive or massless. We'll start with the massive case first, as a warmup for the massless case. Physically, this could be referring to the Z^μ massive vector bosons of the broken electroweak force.

For the massive vector mesons, write down the general lagrangian:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A^\nu\partial_\mu A^\nu + a\partial_\mu A^\mu\partial_\nu A^\nu + bA_\mu A^\mu).$$

The sign is chosen so that the kinetic terms of the spatial components have the right sign. Write the EOM:

$$-\partial^2A_\nu - a\partial_\nu(\partial\cdot A) + bA_\nu = 0,$$

and note plane wave solutions $A_\mu(x) = \epsilon_\nu e^{-ik\cdot x}$ solves it if $k^2\epsilon_\nu + ak_\nu(k\cdot\epsilon) + b\epsilon_\nu = 0$. The longitudinal solutions have $\epsilon \propto k$ and have mass $\mu_L^2 = -b/(1+a)$. The transverse have mass $\mu_T^2 = -b$. Can eliminate the uninteresting longitudinal solution by taking $a = -1$ and $b \neq 0$, then write Proca lagrangian in terms of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mu^2A_\mu A^\mu$$