11/23 Lecture outline

⋆ **Reading: Tong chapter 5**

• Recall, the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$, and we considered plane wave solutions

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

and found that these satisfy the Dirac equation provided that $p^2 = m^2$ (good!) and

$$(\gamma^\mu p_\mu - m)u^s(p) = 0, \qquad (\gamma_\mu p^\mu + m)v^r(p) = 0.$$

The important properties are that these form a complete, orthogonal basis, with

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

We'll see how to evaluate Feynman diagrams involving fermions using just these relations. These relations are basis - independent. Explicit expressions for u^r and v^s are less useful and are also basis dependent.

For example, in the Dirac basis:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

in the rest frame of a massive fermion, we get

$$u^{(1)} = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

which can be boosted to get the solution for general p^μ. For the massless case,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and r, s label the basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

• The general solution of the classical EOM is a superposition of these plane waves:

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b^r(p) u^r(p) e^{-ipx} + c^{r\dagger}(p) v^r(p) e^{ipx} \right)$$

The theory is quantized by using $\Pi^0_\psi = \partial\mathcal{L}/\partial(\partial_0\psi) = i\psi^\dagger$ and imposing

$$\{\psi(t,\vec{x}), \Pi(t,\vec{y})\} = i\delta(\vec{x}-\vec{y}), \qquad \text{i.e.} \qquad \{\psi(t,\vec{x}), \psi^\dagger(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}).$$

If we quantize with a commutator rather than anticommutator, get a Hamiltonian that is unbounded below, with c creating antiparticles with negative energy. Shows that spin $\frac{1}{2}$ must have fermionic statistics, to avoid unitarity problems. This is a special case of the general spin-statistics theorem: unitarity requires integer spin fields to be quantized as bosons (commutators) and half-integer spin to be quantized according to Fermi-Dirac statistics (anti-commutators). Leads to the Pauli exclusion principle.

So the coefficients in the plane wave expansion get quantized to be annihilation and creation operators as

$$\{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p}-\vec{p}'), \qquad \{c^r(p), c^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p}-\vec{p}'),$$

with all other anticommutators vanishing.

• Aside on dimensional analysis $[\psi] = 3/2$, $[u] = [v] = 1/2$, $[b] = [c] = -1$.

• Hamiltonian of the Dirac equation, with fermionic statistics, $\mathcal{H} = \Pi_\psi \dot{\psi} - \mathcal{L} = \bar{\psi}(-i\partial_j\gamma^j + m)\psi$, and then $H = \int d^3x \mathcal{H}$ gives

$$: H := \int \frac{d^3p}{(2\pi)^3 2E_p} E_p (b^{r\dagger}(p) b^r(p) + c^{r\dagger}(p) c^r(p)),$$

good, $b^{r\dagger}(p)$ creates a spin 1/2 particle of positive energy, and $c^{r^\dagger}(p)$ creates a spin 1/2 particle of positive energy. The second term was re-ordered according to normal ordering – the terms originally work out to have the opposite order and the opposite sign. Fermionic statistics gives the sign above, upon normal ordering, but Bose statistics would have given the $c^{r\dagger}c^r$ term with a minus sign, leading to H that is unbounded below. We need Fermionic statistics for spin 1/2 fields to get a healthy theory.

• Do perturbation theory as before, but account for Fermi statistics, e.g. $T(\psi(x_1)\psi(x_2)) = -T(\psi(x_2)\psi(x_1))$ and likewise for normal ordered products. Anytime Fermionic variables are exchanged, pick up a minus sign (and sometimes the additional term if the anti-commutator is non-zero). Consider in particular the propagator

$$\{\psi(x), \bar{\psi}(y)\} = (i\partial\!\!\!/_x + m)(D(x-y) - D(y-x)).$$

and the contraction

$$\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}e^{-ip(x-y)}.$$

Vanishes for spacelike separated points. The momentum space fermion propagator is

$$\frac{i}{\not{p}-m+i\epsilon}.$$