11/16 Lecture outline

⋆ **Reading for today's lecture: Coleman lecture notes pages 211-248.**

- Last time: The basic spinor reps are $(1/2,0)$ and $(0,1/2)$, denoted $u_\pm$; these are called left and right handed Weyl spinors. They both have $D=e^{-i\vec\sigma\cdot\hat e\theta/2}$ for a rotation by θ around the $\widehat{e}$ axis, but they have $D_\pm=e^{\pm\vec\sigma\cdot\hat e\phi/2}$ for a boost along the $\widehat{e}$ axis, where $v=\tanh\phi$. These 2-component Weyl spinor representations individually play an important role in non-parity invariant theories, like the weak interactions. Parity $((t,\vec x)\to(t,-\vec x))$ exchances them. So, in parity invariant theories, like QED, they are combined into a 4-component Dirac spinor, $(1/2,0)+(0,1/2)$:

$$\psi=\begin{pmatrix}u_+\\ u_-\end{pmatrix}.$$

The 4-component spinor rep starts with the clifford algebra $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\mathbf{1}$, e.g.

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}.$$

There are other choices of reps of the clifford algebra. The $\mathcal{M}^{\mu\nu}$ rotation / boost generators become (the extension / analog of spin $\vec S$ for the 3d rotation group) $S^{\mu\nu}=\frac14[\gamma^\mu,\gamma^\nu]=\frac12\gamma^\mu\gamma^\nu-\frac12\eta^{\mu\nu}$, satisfies the Lorentz Lie algebra relation. Under a rotation, $S^{ij}=-\frac{i}{2}\epsilon_{ijk}\begin{pmatrix}\sigma^k&0\\0&\sigma^k\end{pmatrix}$, so taking $\Omega_{ij}=-\epsilon_{ijk}\varphi^k$ get under rotations

$$S[\vec\varphi]=\begin{pmatrix}e^{i\vec\varphi\cdot\vec\sigma/2}&0\\0&e^{i\vec\varphi\cdot\vec\sigma/2}\end{pmatrix}.$$

Under boosts, $\Omega_{i,0}=\phi_i$,

$$S[\Lambda]=\begin{pmatrix}e^{\vec\phi\cdot\vec\sigma/2}&0\\0&e^{-\vec\phi\cdot\vec\sigma/2}\end{pmatrix}.$$

This exhibits the 2-component reps that we described above.

- Under Lorentz transformations, spinors transform as $\psi(x)\to S[\Lambda]\psi(\Lambda^{-1}x)$, and $\psi^\dagger(x)\to\psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger$. Note that $S[\Lambda]^\dagger S[\Lambda]\neq 1$, but $S[\Lambda]^\dagger=\gamma^0S[\Lambda]^{-1}\gamma_0$. So define $\bar\psi(x)\equiv\psi^\dagger\gamma^0$ and note that $\bar\psi\psi$ transforms as a scalar, and $\bar\psi\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

For 2-component spinors, $u_-^\dagger\sigma^\mu u_-$ and $u_+^\dagger\bar\sigma^\mu u_+$ transform like vectors, where $\sigma^\mu=(1,\sigma^i)$ and $\bar\sigma^\mu=(1,-\sigma^i)$. Here are two Lorentz scalars (exchanged under parity): $u_\pm^\dagger u_\mp$.

$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$, anticommutes with all other γ^μ and $(\gamma^5)^2 = 1$. In our above representation of the gamma matrices, $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are projection operators, projecting on to $u_\pm$.

- The Dirac action:

$$\begin{aligned} S &= \int d^4x \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \\ &= \int d^4x (u_+^\dagger i\sigma^\mu\partial_\mu u_+ + u_-^\dagger i\bar{\sigma}^\mu\partial_\mu u_- - m(u_+^\dagger u_- + u_-^\dagger u^+)). \end{aligned}$$

The last line exhibits something interesting: if there is a mass term, it is necessary to have both u_R and u_L (preserving parity, which takes $\vec{x} \to -\vec{x}$ and exchanges $u_R \leftrightarrow u_L$). If $m = 0$, we can consider P non-invariant theories with only u_R or only u_L. More about this perhaps in a later quarter. Also, the action has a global $U(1)$ symmetry under $\psi \to e^{i\alpha}\psi$, whose Noether conserved charge is fermion number. If $m = 0$, this symmetry is enhanced to $U(1)_R \times U(1)_L$, acting separately on u_R and u_L. Neat point: this enhanced symmetry helps explains why the known fermion masses are small. Call $U(1)_V \cong U(1)_R + U(1)_L$ and $U(1)_A \cong U(1)_R - U(1)_L$. Also, call $u_R \equiv u_+$ and $u_L \equiv u_-$.

Starting from the above Lagrangian, we get the EL equations from minimizing the action. Vary $\mathcal{L}$ w.r.t. $\bar{\psi}$ to get the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Dirac wrote this down by thinking about how to make sense of the square-root of the operator appearing in the KG equation, $\sqrt{\partial_\mu\partial^\mu + m^2}$; indeed, $-(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m) = \partial^2 + m^2$.

The conjugate momentum to ψ is

$$\pi_\psi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu.$$

So ψ has 4 (rather than 8) real d.o.f., it is the phase-space that has 8 d.o.f.

Let's first consider the plane wave solutions for a single Wey spinor u_+, in the $m = 0$ case, so the EOM is $\partial_\mu\sigma^\mu u_+(x) = 0$. Take positive energy, $k_0 = +\sqrt{\vec{k}^2}$, and then the plane wave solutions are

$$u_+(x) = u_+ e^{-ikx}, \qquad \text{or} \qquad u_+(x) = v_+ e^{ikx}.$$

When we quantize, u_+ will go with a particle annihilation operator, and v_+ will go with an antiparticle creation operator. Plugging into the EOM, $(k_0 - \vec{\sigma}\cdot\vec{k})u_+ = 0$. Taking $\vec{k} = k_0\hat{z}$, get

$$\langle 0|u_+(x)|k\rangle \propto e^{-ikx}\begin{pmatrix}1\\0\end{pmatrix}.$$

Note also that the state $|k\rangle$ has spin $J_z = 1/2$, under a rotation by θ around the $\widehat{z}$ axis, it picks up a phase $e^{i\theta/2}$. The state $|k\rangle$ thus carries helicity $+1/2$, and the annihilation operator that goes with u_+ annihilates that state. Likewise

$$\langle k|v_+(x)|0\rangle \propto e^{ikx}\begin{pmatrix}1\\0\end{pmatrix},$$

so v_+ goes with a creation operator creating states of angular momentum $-1/2$ along the direction of motion, i.e. helicity $-\frac{1}{2}$. The theory has particles of helicity $1/2$ and antiparticles of helicity $-1/2$. This can only happen for massless fermions, since otherwise could get the opposite helicity in a boosted frame. Neutrinos are like this. All neutrinos are "left-handed". Nice story from Feynman Lectures on Physics about shaking hands with an alien. (At the time the story was written, it was known that C and P are separately broken, but thought that CP was a valid symmetry. CP would be a valid symmetry if there were only two matter generations in the Standard Model. Now we know that there are actually three generations, and that CP is also violated, by tiny effects. Lorentz symmetry implies that CPT is a valid symmetry, so CP violation is equivalent to breaking of time reversal symmetry at the microscopic level.)

The plane wave solutions of the Dirac equation are

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

where

$$(\gamma^\mu p_\mu - m)u^s(p) = 0, \qquad (\gamma_\mu p^\mu + m)v^r(p) = 0.$$

If we wanted to solve the eigenvalue equation $\gamma_\mu p^\mu X = \lambda X$, we'd find four eigenvalues, and four linearly independent eigenvectors, which form a basis. Here, because $\not{p}^2 = m^2$, we see that $\lambda = \pm m$, so there are two eigenvectors with $\lambda = m$, i.e. u^s, and two with $\lambda = -m$, i.e. v^r. Here r, s both run over $1, 2$, labeling the four eigenvectors, each of which is a 4-component vector.

The important properties are that these form a complete, orthogonal basis, with

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

We'll see how to evaluate Feynman diagrams involving fermions using just these relations. These relations are basis - independent. Explicit expressions for u^r and v^s are less useful and are also basis dependent.

For example, in the Dirac basis:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

in the rest frame of a massive fermion, we get

$$u^{(1)} = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

which can be boosted to get the solution for general p^μ. For the massless case,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and r, s label the basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.