215a Homework exercises 3, Fall 2015, due Oct. 21

1. Relativity review. In lecture, we saw that $\int d^4p\delta(p^2 - m^2) \to d^3\vec{k}/2\omega$, where $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$, which shows that $d^3\vec{p}/2\omega_{\vec{p}}$ is Lorentz invariant. Verify explicitly that this quantity is invariant under Lorentz boosts along the x-axis, using the fact that $(\omega, \vec{p})$ transforms as a 4-vector.

2. In lecture, we used Fourier transforms to construct Greens functions for the Klein-Gordon equation. In this exercise, you will explicitly verify that

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2\right)\langle 0|T(\phi(x)\phi(y))|0\rangle = C\delta^4(x-y),$$

and determine the constant C. Don't use the Fourier transforms from lecture. Instead just use the KG field equation for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, the equal time commutation relations, and the definition of the time ordered product T. Hint: use the definition of T involving $\theta(x^0 - y^0)$ and $\theta(y^0 - x^0)$, and the fact that the derivative of the theta function gives a delta function.

This exercise illustrates that equations of motion, which are operator equations, don't necessarily give zero in time ordered correlation functions – instead, it can give contact interactions when the operator points coincide. We'll soon see similar effects with current conservation laws, which can also have specific contact interactions when $\partial_\mu j^\mu(x)$ is in a correlation function (these are called Ward identities).

3. Consider a real scalar field with ϕ^4 interaction: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$.

(a) What is the mass dimension of $\phi(x)$, and of λ, if the above is the Lagrangian density in D spacetime dimensions? Write it as $[\lambda] = n$ if $\lambda \sim (\text{mass})^n$. Recall $[S] = 0$ and $S = \int d^Dx\mathcal{L}$. Your answers to all these questions will depend on D. (This is essentially a repeat of one of the earlier HW exercises, but it emphasizes that we can assign mass dimension to parameters in the Lagrangian, and scale symmetry is broken if they have non-zero mass dimension.)

(b) Write $\phi(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2E_k}(a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ikx})$. We'll say that a and $a^\dagger$ annihilate and create "mesons." Using your result from above, what is the mass dimension $[a(k)]$ and $[a^\dagger(k)]$? As a check, it should give the answer found in class for $D = 4$.

(c) Define $\langle f|S|i\rangle = i\mathcal{A}_{fi}(2\pi)^D\delta^D(p_f - p_i)$. Suppose that $|i\rangle$ has n_i mesons and $\langle f|$ has n_f mesons. What is the mass dimension $[|i\rangle]$ and $\langle f|]$ and $[\mathcal{A}_{fi}]$? Again, verify that it reduces to the answer given in class for $D = 4$.

(d) Find the amplitude (with D general) for the decay $\phi \to \phi + \phi + \phi$, to leading order in λ. Verify that your result is consistent with dimensional analysis and the above results. **Note:** the interaction term $: \mathcal{H}_I :$ should be taken to be normal ordered, so there is no contribution involving a loop. Without the normal ordering, there would have been a term where e.g. the initial p_1^μ is equal to one of the outgoing q_i^μ, and the other two outgoing particles are connected to ϕ^2 in the interaction with the other ϕ^2 in the interaction connecting to each other. This is a term that is eliminated by normal ordering. As we'll discuss later, this is a special case of what is called counter terms.