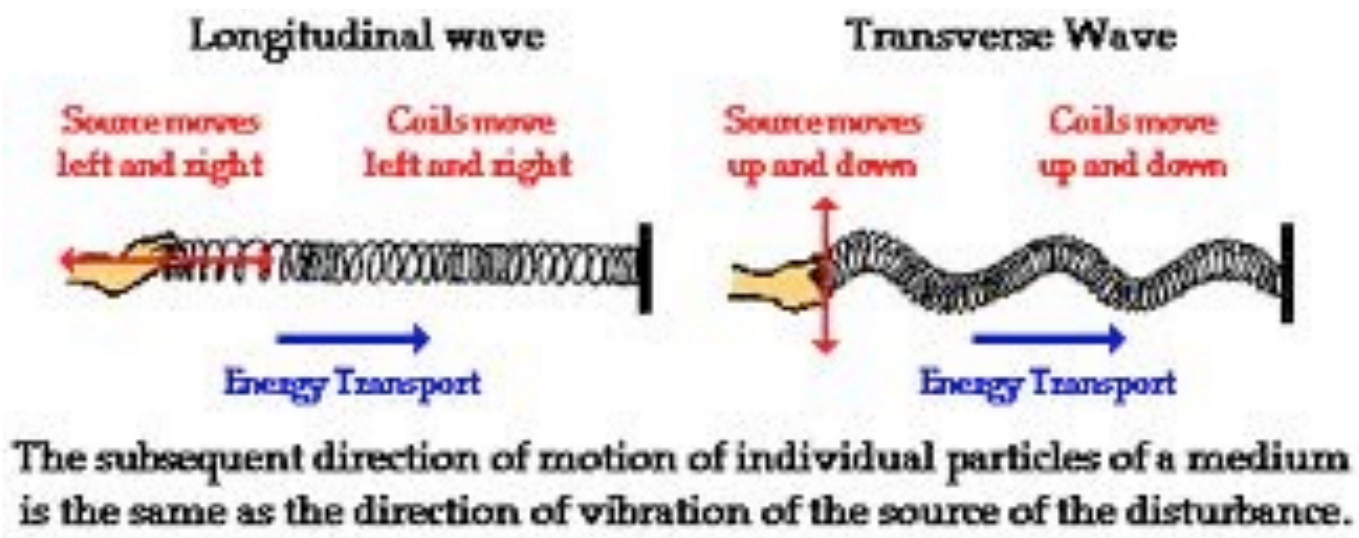
Mechanical Waves

Ken Intriligator's week 2 lectures, Oct 7, 2013

Traveling wave

Standing wave

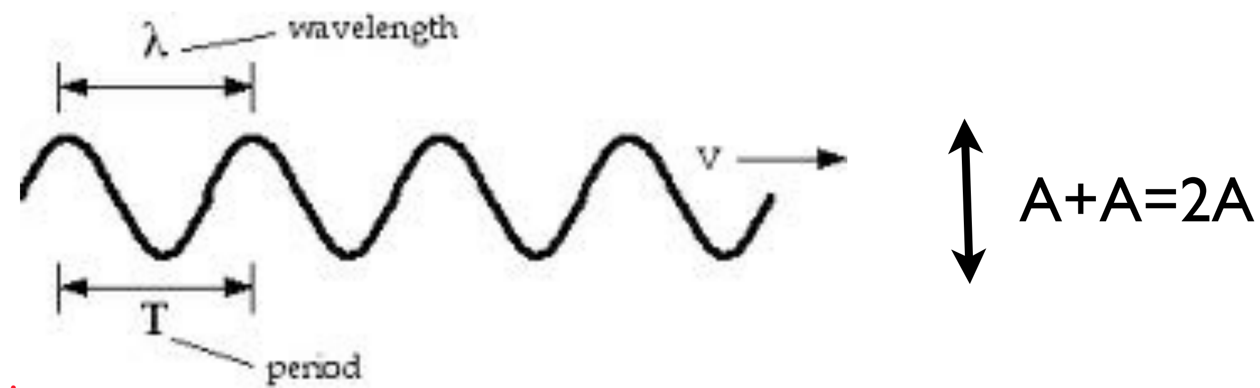
Osc. direction vs energy transport dir.



E.g. earthquakes: primary (fastest traveling) wave is longitudinal, secondary (slower traveling) wave is transverse. The second is often more damaging.

Periodic waves

In **both** space and time.



$$\omega = 2\pi/T \qquad k = 2\pi/\lambda$$

ω: (angular) frequency; T: period; k: wave number; λ: wave length

Like SHO, A is the amplitude

$$y(t, x=0) = A\cos(\omega t)$$

$$y(t=0, x) = A\cos(kx)$$

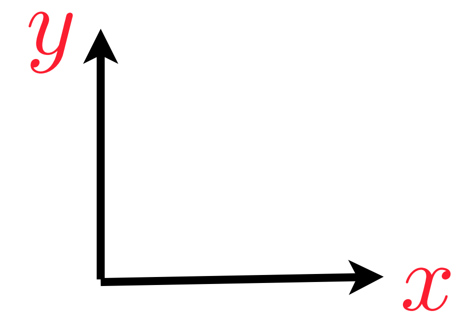
$$y(t+T, x) = y(t, x)$$
$$y(t, x+\lambda) = y(t, x)$$

e.g.

$$y(t, x) = A\cos(kx - \omega t) \longleftarrow \text{Traveling wave}$$

$$y(t, x) = A\cos(kx)\cos(\omega t) \longleftarrow \text{Standing wave}$$

Traveling wave case



Right moving: $y(t,x) = A\cos(kx - \omega t)$

Left moving: $y(t,x) = A\cos(kx + \omega t)$

More gen'ly: $\psi(\vec{x},t) = A\cos(\vec{k}\cdot\vec{x} - \omega t)$ $|\vec{k}| = 2\pi/\lambda$

Vector k points in the dir. the wave (its energy) is going.

Phase velocity (speed) of wave: $d(kx - \omega t) = kdx - \omega dt = 0$

$$\longrightarrow \boxed{v_p = \frac{dx}{dt} = \frac{\omega}{k} = \frac{\lambda}{T}}$$

Aside: "group velocity"

For later, general case: $\omega = \omega(\vec{k})$ "dispersion relation"

Dispersion rel'n function depends on the wave type and medium.

E.g. deep water waves: $\omega_{deep\ water} \approx \sqrt{gk}$

$$v_{phase} \equiv \frac{\omega}{k}$$

$$v_{group} \equiv \frac{d\omega}{dk}$$

We'll discuss the physical distinction between them **later.**

D.A. quiz question

MASS, LENGTH, TIME

Want to make a velocity, using only g, lambda and maybe the density rho. Velocity has units of length over time. Lambda has units of length g has units of length over time-squared. rho has units of mass over length-cubed. The units do not allow rho to enter, since no way to cancel its mass. Velocity units can be obtained only as

$$v \sim \sqrt{g\lambda} \sim \sqrt{\frac{g}{k}} \quad \longrightarrow \quad (v_2/v_1) = \sqrt{g_2\lambda_2/g_1\lambda_1}$$

Standing waves

=Superposition of left + right moving wave

Here, person makes right moving wave, and the B.C. at the other end reflects it back, total is standing wave

$$A\cos(kx-\omega t)+A\cos(kx+\omega t)=2A\cos(kx)\cos(\omega t)$$

To the right + To the left = "Standing"

useful trig. identities:

$$\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$$

$$\sin(a\pm b)=\sin a\cos b\pm\cos a\sin b$$

Wave equation

1d: $(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\psi(t,x) = 0$

Linear 2nd order PDE

:-)! Nice! Superposition!

DA: Same units. Correct!

3 d: $(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi(t,x) = 0$

We'll discuss 3d case later.
This week, just 1d waves.

$$\psi(t,x) = A\cos(k(x - vt))$$

$$\psi(t,x) = A\cos(k(x + vt))$$

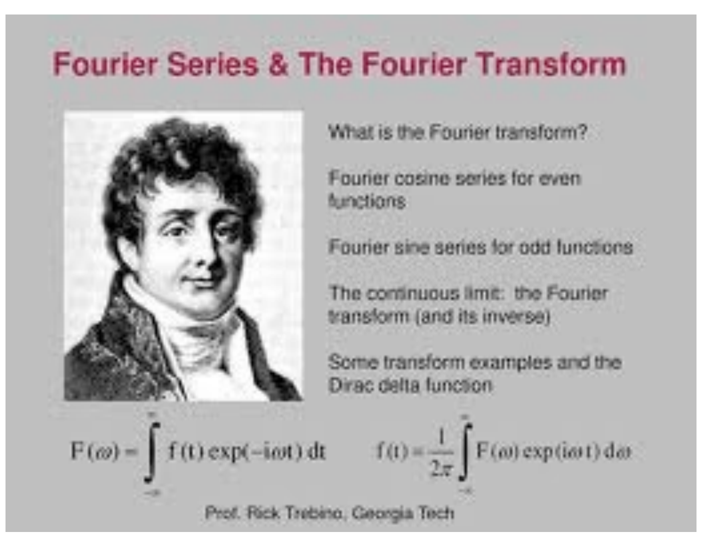
$$\psi(t,x) = A\cos(kx)\cos(kvt)$$

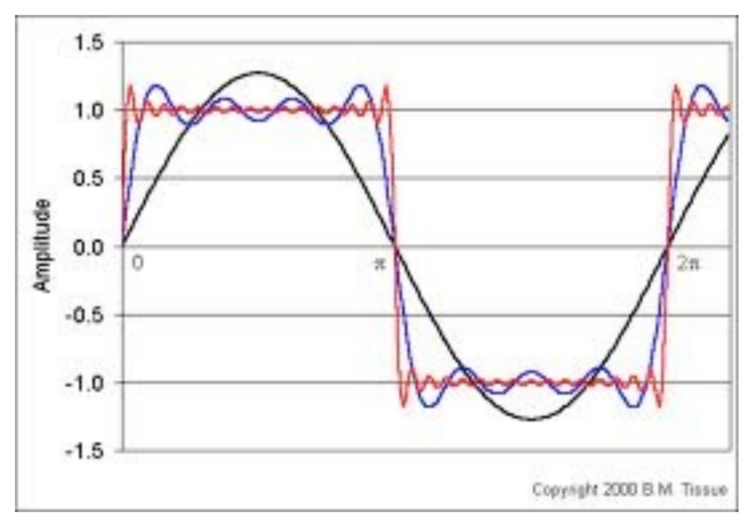
Examples solutions of the 1d wave equation. Superpose for general solution (Fourier).

(Aside: Fourier)

Math statement: get general functions from a sum (superposition) of sin or cos functions.

Physics application: get general solution of the wave equation from a superposition of waves of definite frequency and wavelength





Wave equation, cont.

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi(t,x) = 0$$

Is solved by: $\psi = \psi_R(x - vt) + \psi_L(x + vt)$

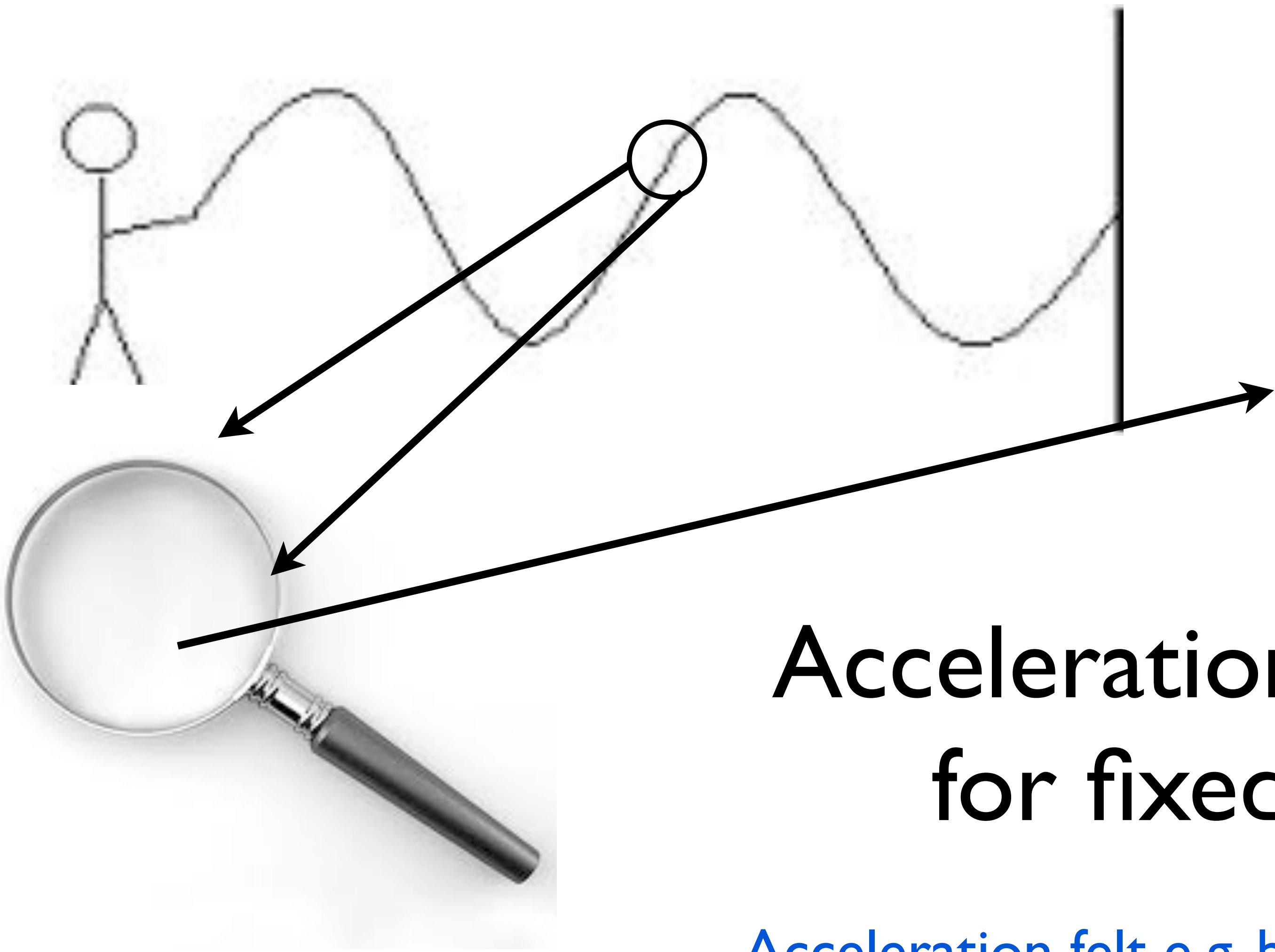
Arbitrary functions for right and left moving parts.

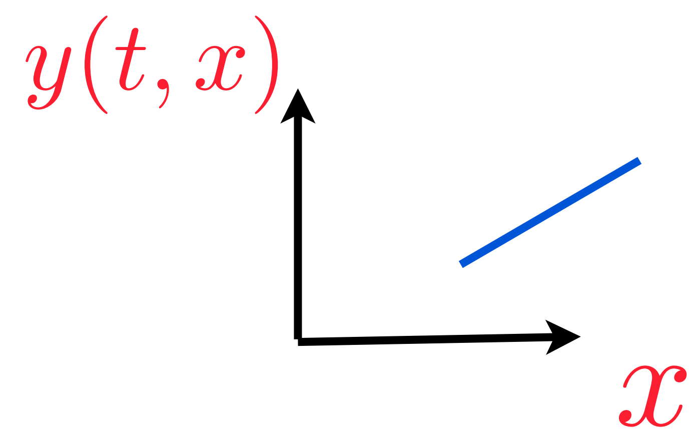
E.g. right moving $y(t,x) = A\cos(kx - \omega t)$

Velocity (speed) is the phase velocity:

$$v = \frac{\omega}{k} = \frac{\lambda}{T}$$

Waves on a string





Acceleration in y direction, for fixed position x: $\frac{\partial^2 y}{\partial t^2}$

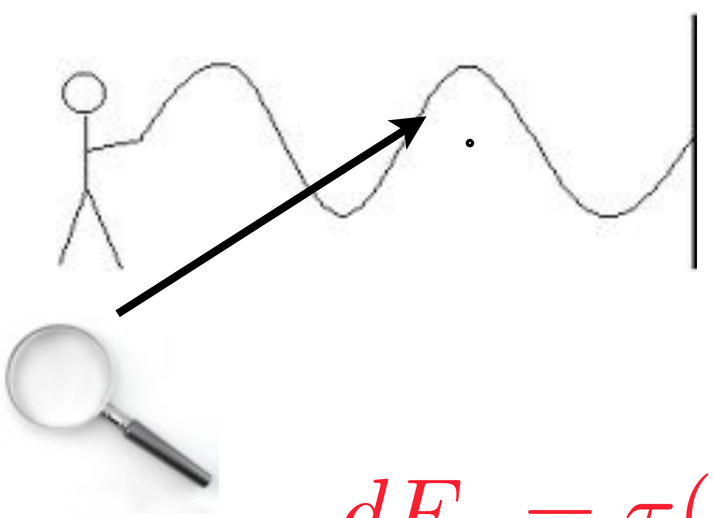
Acceleration felt e.g. by an ant at position x.

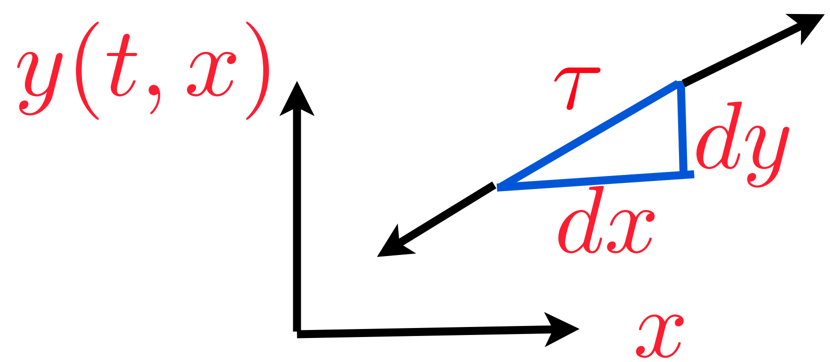
Segment of string from x to x+dx has mass: μdx

Linear mass density of the string

Derive wave eqn.

Follows from F=ma, applied to string elements.





$$dF_y = \tau\left(\frac{\partial y}{\partial x}\Big|_{x+dx} - \frac{\partial y}{\partial x}\Big|_x\right) \longrightarrow dF_y = \tau\frac{\partial^2 y}{\partial x^2}dx$$

Tension ("F" in book).

$$dF_y = (dm)a_y = (\mu dx)\frac{\partial^2 y}{\partial t^2}$$

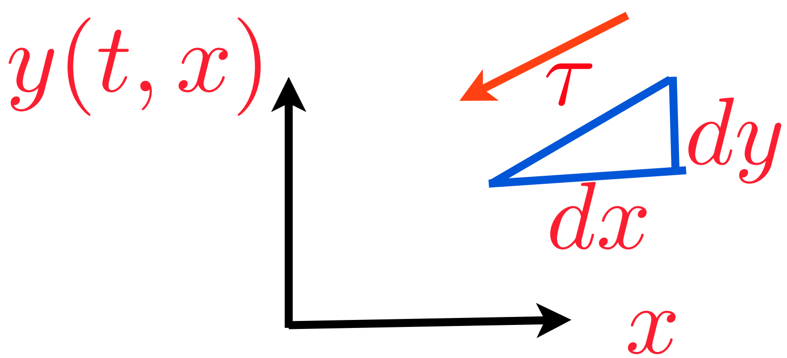
Linear mass density

Equating gives the 1d wave eqn, with

$$v = \sqrt{\frac{\tau}{\mu}}$$

Ex: verify the units work, DA!

Wave energy, power



Force exerted on string to the right, by the string to the left

Method 1:

$$P(x,t) = \vec{F}(x,t)\cdot\vec{v}(x,t) = F_y(x,t)v_y(x,t) = -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

Method 2: $k(x,t) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \qquad u(x,t) = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$ energy densities

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = (k+u)v$$

Both methods give the same answer (using the wave eqn):

Wave power, cont.

$$P(x,t) = \vec{F}(x,t) \cdot \vec{v}(x,t) = F_y(x,t)v_y(x,t) = -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

$$y = A\cos(kx - \omega t)$$

$$P = \tau k \omega A^2 \sin^2(kx - \omega t) \qquad \tau k \omega = \frac{\tau}{v}\omega^2 = \sqrt{\tau\mu}\,\omega^2$$

$$P_{max} = \sqrt{\tau\mu}\,\omega^2 A^2$$ and $$P_{ave} = \frac{1}{2}\sqrt{\tau\mu}\,\omega^2 A^2$$

$$P_{ave} = \frac{1}{2}P_{max}$$ since $$\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = \frac{1}{2}$$

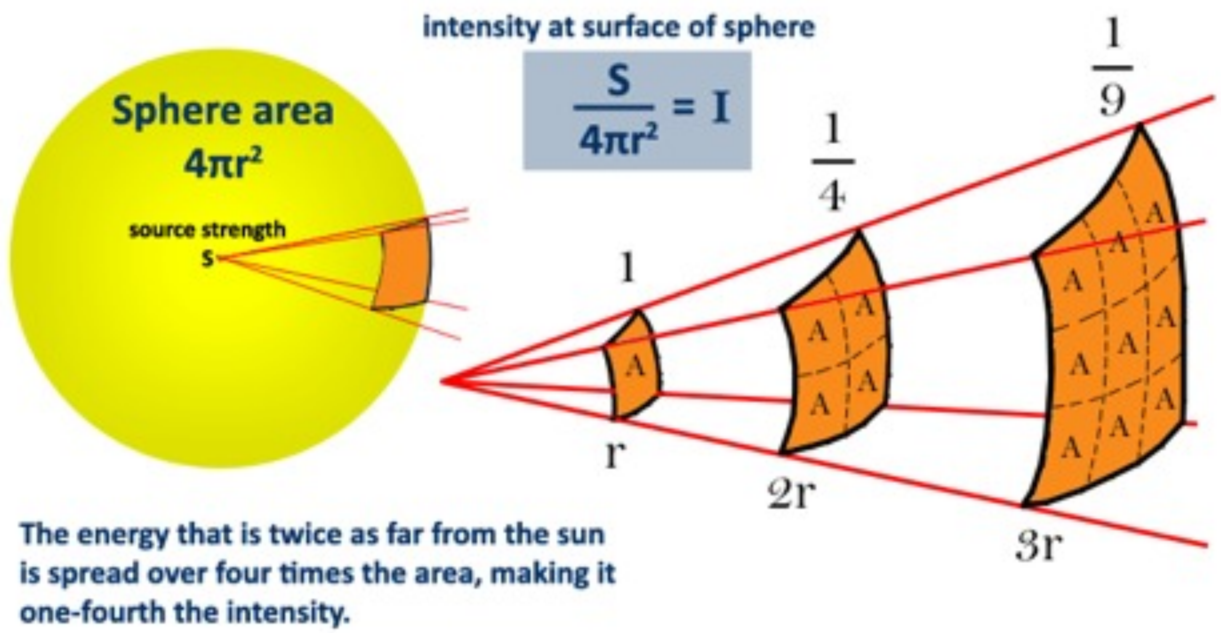
3d wave, intensity

Intensity = I = energy flux, i.e. energy flow per area, per time. In 3d, I drops as distance r-squared, since energy is conserved and area grows as r-squared.

$$I = P/4\pi r^2$$

$$I_1/I_2 = r_2^2/r_1^2$$

E.g. 60W bulb emits P=60W, intensity of the light drops as 1/r^2, since light spreads out.



Play / tune your guitar

Fingers shorten string length, shorter length = higher frequency.

Bass: fatter + longer strings = lower frequency.

Tune: tighten the knobs (increase tension) to get higher frequency.

String waves, modes

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y(t,x) = 0.$$

$n = 3$

Ends, BCs: $y(t,0) = y(t,L) = 0$

Ends, and nodes, have y(t)=0.

$$L = n\frac{1}{2}\lambda$$

$n = 1, 2, 3, 4, \ldots$ ="mode number"

Fundamental mode (n = 1); higher harmonics (n = 2, 3, 4, …)

The wave eqn + B.C. solution:

$$y(t,x) = A\sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

$$\left(k_n \equiv \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}\right)$$

$$\omega_n = v k_n = n\omega_1 \qquad \omega_1 = \sqrt{\frac{\tau}{\mu}}\frac{\pi}{L}$$

Tune your guitar! More bass (lower freq) from fatter or longer strings. Higher freq. from more tension.

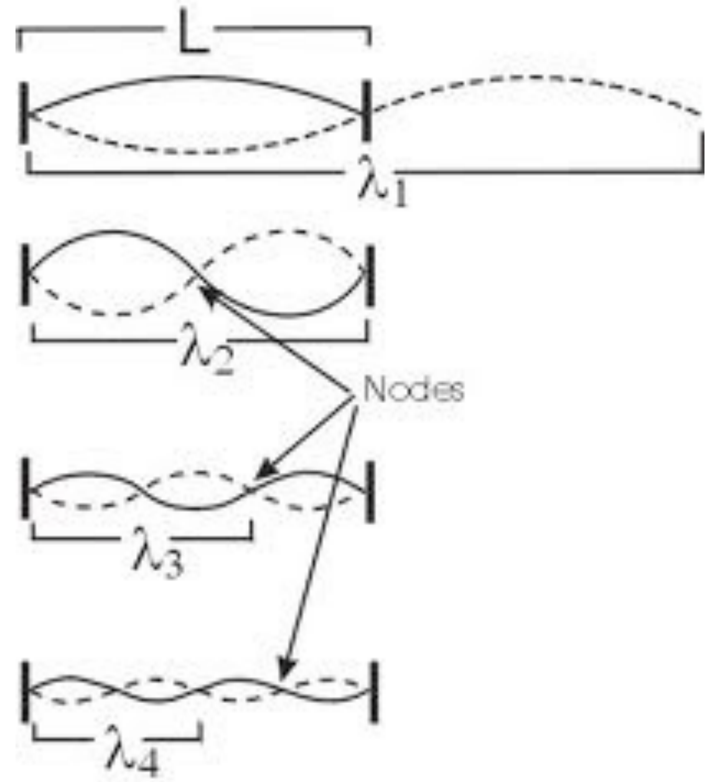
Harmonics

n=1 "fundamental"

n=2, (1 node)

n=3 (n-1 nodes)

n=4



$$\lambda_n = 2L/n$$

$$\left(k_n \equiv \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}\right)$$

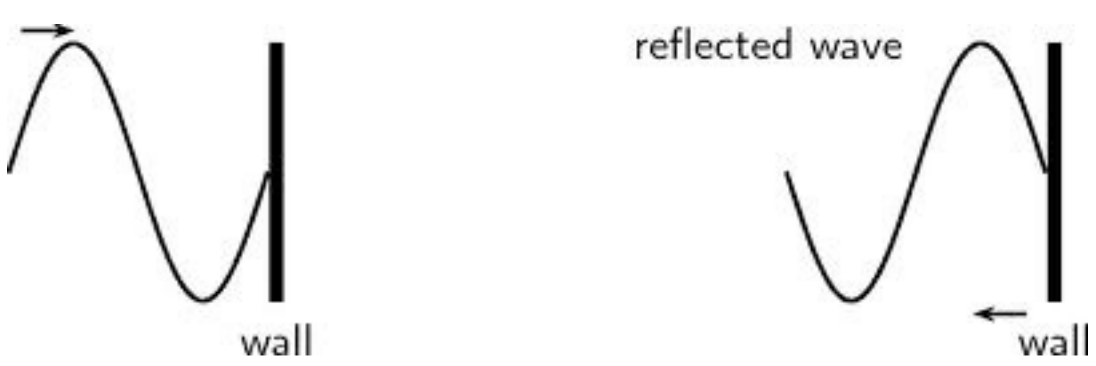
$$\omega_n = vk_n = n\omega_1 \qquad \omega_1 = \sqrt{\frac{\tau}{\mu}}\frac{\pi}{L} \qquad \omega \equiv 2\pi f \equiv 2\pi\nu$$

Wave B.C.s at the ends:

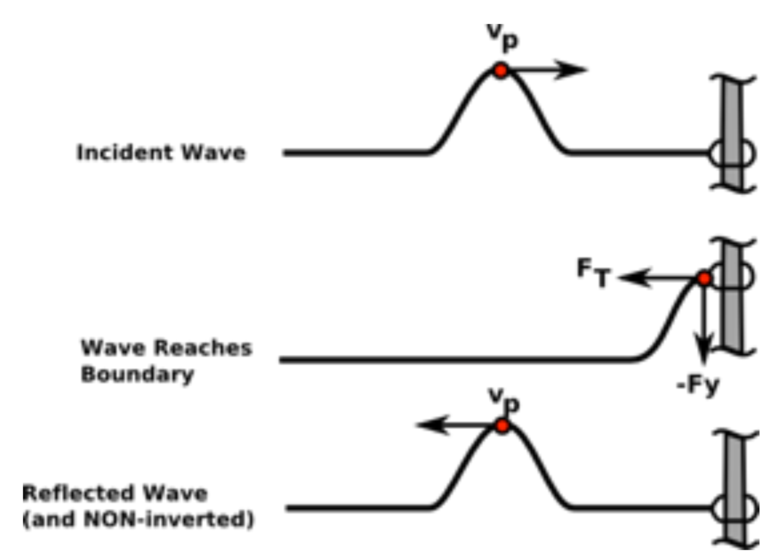
Two common choices:

Fixed (Dirichlet): $y(t, x_{end}) = 0$

Free (Neumann): $\frac{\partial y}{\partial x}(t, x_{end}) = 0$



Fixed (Dirichlet)



Free (Neumann)

Free ends case, sol'n:

Suppose free ends at x=0, and x=L

$$\frac{\partial y}{\partial x}(t,0) = \frac{\partial y}{\partial x}(t,L) = 0$$

$$y = A\cos(\frac{n\pi x}{L})\cos(\frac{n\pi vt}{L})$$

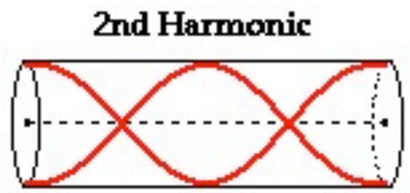
Fixed ends (few slides ago): this cos was instead sin.

The harmonics are similar in the two cases.

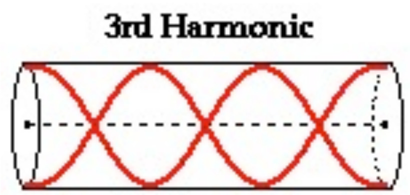
Free ends harmonics

Slope =0 at ends

$$\lambda = 2L$$



$$\lambda = L$$



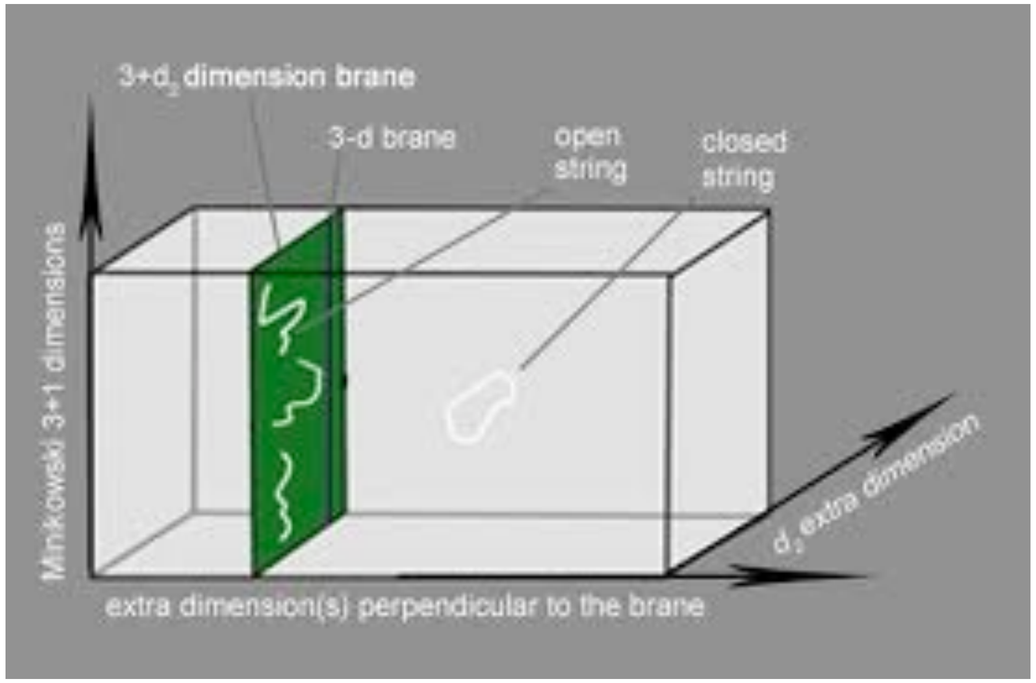
$$\lambda = 2L/3$$

...

$$\lambda_n = 2L/n$$

$$f_n = T_n^{-1} = v/\lambda_n = nv/2L$$

Just for fun: string theory!



Similar wave equation. Different harmonics are different particles. Known particles are the fundamental harmonic, others would be new particles.