10/10 Lecture outline

⋆ **Reading for today's lecture: Luke p. 65-80; Tong p. 35-41.**

• Last time: quantize the KG field theory in $3+1$ dimensions. Write

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} [a_{\vec{k}} e^{-ikx} + a^\dagger_{\vec{k}} e^{ikx}],$$

$$\Pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{k}}}{2}} [a_{\vec{k}} e^{-ikx} - a^\dagger_{\vec{k}} e^{ikx}],$$

Then canonical quantization implies that

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}'),$$

creation and annihilation operators, with others vanishing. It will be useful to define $a(k) \equiv \sqrt{2\omega_k} a_{\vec{k}}$, so then the above becomes

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} [a(k) e^{-ikx} + a^\dagger(k) e^{ikx}],$$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'),$$

with the relativistic-invariant measures, creation and annihilation operators appearing. The quantum field is a superposition of creation and annihilation operators.

• Define (following Heisenberg and Pauli's backward looking notation)

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

with ϕ^+ the term in the FT with the annihilation operators (positive frequency) and ϕ^- the term with the creation operators (negative frequency).

Define normal ordering $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum. So $: \phi^+(x)\phi^-(y) := \phi^-(y)\phi^+(x)$. We'll often define quantities to be normal ordered, e.g. we take

$$H \equiv: H : \int \frac{d^3k}{(2\pi)^2(2\omega)} \omega a^\dagger(\vec{k}) a(\vec{k}),$$

$$\vec{P} \equiv: \vec{P} := \int \frac{d^3k}{(2\pi)^2(2\omega)} \vec{k} a^\dagger(\vec{k}) a(\vec{k}),$$

where we're dropping the CC contributing term in H, as discussed last time. So $P^\mu|0\rangle = 0$ and $P^\mu|p_1\dots p_n\rangle = p^\mu_{tot}|p_1\dots p_n\rangle$, where $|p_1\dots p_n\rangle = \prod_n a^\dagger(k_n)|0\rangle$ and $p^\mu_{tot} = \sum_n p^\mu_n$.

• Causality? Compute $[\phi(x),\phi(y)] = D(x-y) - D(y-x)$, where

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note that the commutator is a c-number, not an operator. Note also that $2i\partial_{x^0}D(x-y)$ is the integral that we saw in last lecture, for the probability amplitude to find a particle having traveled with spacetime displacement $(x-y)^\mu$. For spacelike separation, $(x-y)^2 = -r^2$, we here get $D(x-y) = \frac{m}{2\pi^2 r}K_1(mr)$, with K_1 a Bessel function. Recall that the Bessel function has a simple pole when its argument vanishes, and exponentially decays at infinity. Although $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone, the above difference is not: for spacelike separation, $D(x-y) - D(y-x) = 0$. Good. It's non-vanishing for timelike separation.

• Get more interesting theories by adding interactions, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$, treat 2nd term as a perturbation. We can consider perturbative solutions in both classical or quantum field theory. The starting point is the green's function for the theory with a forcing function source:

• Consider $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - \rho\phi$, where ρ is a classical source. Solve by $\phi = \phi_0 + i\int d^4y D(x-y)\phi(y)$, where ϕ_0 is a solution of the homogeneous KG equation and the green's function $D(x-y)$ satisfies

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y).$$

By a F.T., get

$$D_?(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2+m^2}$. There are choices about whether the contour goes above or below the poles, and that's what the ? label indicates.

Ended here. Continue next time:

Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. Considering $x_0 > y_0$, get that

$$D_R(x-y) = \theta(x_0-y_0)\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)})$$
$$\equiv \theta(x_0-y_0)(D(x-y) - D(y-x)) = \theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle,$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik(x-y)}.$$

This is reasonable: then the $\rho(y)$ source only affects $\phi(x)$ in the future.

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator. Define

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle \phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle \phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\ldots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f|t_i\rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. We'll see that this ensures causality.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator. Let e.g. $\phi(x) = \phi^+(x) + \phi^-(x)$, where ϕ^+ is the term with annihilation operators and ϕ^- is the one with creation operators (using Heisenberg and Pauli's reversed-looking notation). Then for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.

• Simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• In QM we can use the S-picture, $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi\rangle$, or the H-picture, $i\hbar\frac{d}{dt}\mathcal{O}(t) = [\mathcal{O}, H]$. In interacting theories, it is useful to use the hybrid, interaction picture. Write $H = H_0 + H_{int}$.

we use H_0 to time evolve the operators, and H_{int} to time evolve the states:

$$i\frac{d}{dt}\mathcal{O}(t) = [\mathcal{O}, H_0], \qquad i\frac{d}{dt}|\psi(t)\rangle = H_{int}|\psi(t)\rangle.$$

For example, we'll take H_0 to be the free Hamilton of KG fields, with only the mass terms included in the potential. Again, this is free because the EOM are linear, and we can solve for $\phi(x)$ by superposition. As before, upon quantization, the fields become superpositions of creation and annihilation operators. The states are all the various multiparticle states, coming from acting with the creation operators on the vacuum.

• Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Derive it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does.

Now use Wick's theorem:

$$T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + \sum_{contractions} : \phi_1 \ldots \phi_n :$$

to get rid of the time ordered products. (Prove Wick's theorem by iteration.) Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3x g\phi\psi^\dagger\psi$. Use $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$.

Now consider $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ thus involves

$$\langle p_1' p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1 p_2\rangle = \langle p_1' p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

$$= \left(e^{i(p_1' x_1 + p_2' x_2)} + e^{i(p_1' x_2 + p_2' x_1)}\right)\left(e^{-i(p_1 x_1 + p_2 x_2)} + e^{-i(p_1 x_2 + p_2 x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1 - p_1')^2 - \mu^2} + \frac{1}{(p_1 - p_2')^2 - \mu^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2+m^2}, e\widehat{e})$ and $p_2 = (\sqrt{p^2+m^2}, -p\widehat{e})$, $p_1' = (\sqrt{p^2+m^2}, p\widehat{e'})$, $p_2' = (\sqrt{p^2+m^2}, -p\widehat{e'})$, where $\widehat{e}\cdot\widehat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1-\cos\theta)+\mu^2} + \frac{1}{2p^2(1+\cos\theta)+\mu^2}\right).$$

The scattering by ϕ exchange leads to an attractive Yukawa potential. Indeed, the first term in the above amplitude gives, upon using $(p_1 - p_1')^2 - \mu^2 = |\vec{p}_1 + \vec{p}_1'|^2 + \mu^2$, and the Born approximation in NRQM, $\mathcal{A}_{NR} = -i\int d^3\vec{r}e^{-i(\vec{p}'-\vec{p})}U(\vec{r})$, the attractive Yukawa potential

$$U(r) = \int \frac{d^3p}{(2\pi)^3}\frac{-g^2 e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2} = -\frac{g^2}{4\pi r}e^{-\mu r}.$$

This gives Yukawa's explanation of the attraction between nucleons, from meson exchange.