11/14 Lecture outline

⋆ **Reading for today's lecture: Luke chapter VIII**

• Last time: define Green functions or correlation functions by

$$G^{(n)}(x_1, \dots x_n) = \langle\Omega|T\phi_H(x_1)\dots\phi_H(x_n)|\Omega\rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian. Started to show that

$$G^{(n)}(x_1 \dots x_n) = \frac{\langle 0|T\phi_{1I}(x_1)\dots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

where $|0\rangle$ is the vacuum of the free theory, and ϕ_{iI} are interaction picture fields, and the S in the numerator and denominator gives the interaction-Hamiltonian time evolution from $-\infty$ to x_n, then from x_n to x_{n-1} etc and finally to $t = +\infty$. To show it, take $t_1 > t_2 \dots > t_n$ and put in factors of $U_I(t_a, t_b) = T\exp(-i\int_{t_a}^{t_b} H_I)$ to convert ϕ_I to ϕ_H, using $\phi_H(x_i) = U_I(t_i, 0)^\dagger\phi_I(x_i)U_I(t_i, 0)$. Get $\langle 0|U_I(\infty, t_1)\phi_H(t_1)\dots\phi_H(t_n)U_I(t_n, -\infty)|0\rangle$, and U_I at ends can be replaced with full $U(t_1, t_2)$, since $H_0|0\rangle = 0$ anyway. Now use

$$\begin{aligned}\langle\Psi|U(t, -\infty)|0\rangle &= \langle\Psi|U(t, -\infty)\left(|\Omega\rangle\langle\Omega| + \sum\int|n\rangle\langle n|\right)|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\sum\int e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle\end{aligned}$$

where 1 was inserted as a complete set of states, including the vacuum and single and multiparticle states, including integrating over their momenta, but the wildly oscillating factor kills all those terms. (Riemann-Lebesgue lemma: $\lim_{t\to\infty}\int d\omega f(\omega)e^{i\omega t} = 0$ for nice $f(\omega)$) The result follows upon doing the same for the denominator.

The $\langle 0|S|0\rangle$ in the denominator eliminates the vacuum bubble diagrams. So we have

$$G^{(n)}(x_1, \dots x_n) = \sum \text{Feynman graphs without vacuum bubbles.}$$

It's often more convenient often to work in momentum space,

$$\widetilde{G}^{(n)}(p_1, \dots p_n) = \int\prod_{i=1}^{n} d^4x_i e^{-ip_i x_i} G^{(n)}(x_1 \dots x_n).$$

Similar to what we computed before to get S-matrix elements, but the external legs include their propagators, and the external momenta are not on-shell.

Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}' - \vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x} Z_\phi^{1/2}.$$

The LSZ formula, in terms of the above Green's functions, is:

$$\langle q_1 \dots q_n|S-1|k_1 \dots k_m\rangle = \prod_{a=1}^{n} \frac{q_a^2 - m_a^2}{i\sqrt{Z}} \prod_{b=1}^{m} \frac{k_b^2 - m_b^2}{i\sqrt{Z}} \widetilde{G}^{(n+m)}(-q_1, \dots - q_n, k_1, \dots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.

To derive the LSZ formula, consider wave packets, with some profile $F(\vec{k})$, and $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2 + \mu^2}$, so $f(x)$ solves the KG equation. Now define

$$\phi^f(t) = i\int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

Note that, since $f(x)$ satisfies the KG equation, can show

$$i\int d^4x f(x)(\partial^2 + \mu^2)\phi(x) = -\int dt \frac{\partial}{\partial t}\phi^f(t) = -\phi^f(t)|_{-\infty}^{\infty}.$$

(Sign choice nice for making in states.) Show that $\phi^f(t)$ makes single particle wave packets from the vacuum, $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$. Can similarly show (because of a relative minus sign), $\langle\Omega|\phi^f(t)|k\rangle = 0$, and $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n} + p_n^0}{2\omega_{p_n}} F(\vec{p}_n) e^{-i(\omega_{p_n} - p_n^0)t}\langle n|\phi(0)|\Omega\rangle$, where $\omega_{p_n} \equiv \sqrt{\vec{p}_n^2 + \mu^2}$, which has $\omega_{p_n} < p_n^0$ for any multiparticle state. So $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. Then show

$$\langle f_m|S-1|f_n\rangle = \int \prod_n d^4x_n f_n(x_n) \prod_m d^4x_m f_m(x_m)^* \prod_r i(\partial_r^2 + m_r^2) G(x_n, x_m).$$

Take $f_i(x) \to e^{-ik_i x_i}$ at the end. Show that all the $t \to \pm\infty$ do the right thing to give the in and out states, thanks to various cancellations, using $\lim_{t\to\pm\infty}\langle\Psi|\phi^f(t)|\Omega\rangle = \langle\Psi|f\rangle$.

$\star$ **NEXT TOPIC. Reading: Luke chapter 9. Tong chapter 4**

• On to fermions! Consider more generally Lorentz transformations. Under lorentz transformations $x^\mu \to x^{\mu'} = \Lambda^\mu_\nu x^\nu$, scalar fields transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. Vector fields transform as $A^\mu \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. Generally, $\phi^a \to D[\Lambda]^a_b \phi^b(\Lambda^{-1}x)$, where $D[\Lambda]$ is a rep of the Lorentz group, $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$.