10/31 Lecture outline

⋆ **Reading for today's lecture: Luke p. 65-80; Tong p. 35-41.**

• Last time: amplitudes in our nucleon + meson toy model, via

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x\mathcal{H}_I(x)}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f-p_i).$$

Examples: $N+N\to N+N$, to $\mathcal{O}(g^2)$:

$$\mathcal{A}=(-ig)^2\left[\frac{1}{(p_1-p_1')^2-\mu^2}+\frac{1}{(p_1-p_2')^2-\mu^2}\right].$$

(1) $N(p_1)+\bar{N}(p_2)\to N(p_1')+\bar{N}(p_2')$ has

$$i\mathcal{A}=(-ig)^2\left(\frac{i}{(p_1-p_1')-\mu^2}+\frac{i}{(p_1+p_2)-\mu^2}\right).$$

(2) $N(p_1)+\bar{N}(p_2)\to \phi(p_1')\phi(p_2')$ has

$$i\mathcal{A}=(-ig)^2\left(\frac{i}{(p_1-p_1')-m^2}+\frac{i}{(p_1-p_2')-m^2}\right).$$

(3) $N(p_1)+\phi(p_2)\to N(p_1')+\phi(p_2')$ has

$$i\mathcal{A}=(-ig)^2\left(\frac{i}{(p_1-p_2')-m^2}+\frac{i}{(p_1+p_2)-m^2}\right).$$

• Mandelstam variables. $s=(p_1+p_2)^2$, $t=(p_1-p_1')^2$, $u=(p_1-p_2')^2$, with $s+t+u=4m^2$ (more generally, $s+t+u=\sum_{i=1}^4 m_i^2$). In CM, $s=4E^2$, $t=-2\vec{p}^2(1-\cos\theta)$, and $u=-2\vec{p}^2(1+\cos\theta)$.

• Crossing symmetry, CPT. Write $1+2\to\bar{3}+\bar{4}$. Take all momenta incoming, $\mathcal{A}(p_1,p_2,p_3,p_4)$, with $p_1+p_2+p_3+p_4=0$ and use $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$ and $u=(p_1+p_4)^2$. Note $s+t+u=\sum_{n=1}^4 m_n^2$. The process $1+2\to\bar{3}+\bar{4}$ is kinematically allowed for $s>4m^2$, $t<0$, $u<0$. If instead $u>4m^2$, it's the process $1+3\to\bar{2}+\bar{4}$.

• Yukawa potential. Indeed, the t-channel term in e.g. the above $N+N$ scattering amplitude gives, upon using $(p_1-p_1')^2-\mu^2=-(|\vec{p}_1-\vec{p}_1'|^2+\mu^2)$, and the Born approximation[1] in NRQM, $\mathcal{A}_{NR}=\int d^3\vec{r}e^{-i(\vec{p}'-\vec{p})}V(\vec{r})$, the attractive Yukawa potential

$$V(r)=\int\frac{d^3q}{(2\pi)^3}\frac{-(g/2m)^2e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2}=-\frac{(g/2m)^2}{4\pi r}e^{-\mu r}.$$

[1] Max Born, in QM, or Lord Rayleigh classically: $\frac{d\sigma}{d\Omega}\sim|U(\vec{q})|^2$

(The $1/(2m)^2$ is because we normalized the relativistic states with the extra factor of $2E \approx 2m$ as compared with standard nonrelativistic normalization[2]. This gives Yukawa's explanation of the attraction between nucleons, from meson exchange. The u-channel term is an exchange potential interaction, which exchanges the positions of the two identical particles in addition to giving a potential. For angular momentum ℓ in a partial-wave expansion, the exchange term differs from the direct one by a factor of $(-1)^\ell$.

• We saw above that the t channel term above is associated with the Yukawa potential. The u channel term is similar. Now consider the s channel, in e.g. the $N + \bar{N}$ scattering amplitude. Using the CM relations $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $E_1 = E_2 = \sqrt{p^2 + m^2}$ gives

$$\mathcal{A} \sim \frac{1}{4m^2 + 4p^2 - \mu^2 + i\epsilon},$$

so for $\mu < 2m$ the denominator is always positive, and the amplitude decreases with increasing p^2. For $\mu > 2m$ there is a pole at $(p_1 + p_2)^2 = \mu^2$, where the intermediate meson goes on shell. This leads to a peak (not a pole, of course; because the intermediate particle is unstable anyway, the denominator gets an imaginary contribution from higher order contributions), a *resonance*, in the cross section. E.g. Z_0 pole in $e^+e^- \to \mu^+\mu^-$, but not in $e^+e^- \to \gamma\gamma$.

• Solve $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - J(x)\phi$. Using Dyson + Wick's theorem, $U(\infty, -\infty) =: e^{O_1 + \frac{1}{2}O_2} :$, where $O_1 = -i\int d^4x J(x)\phi(x)$ and $O_2 = (-i)^2 \int d^4x_1 d^4x_2 D_F(x_1 - x_2) J(x_1) J(x_2)$. So $O_2 = \alpha + i\beta$ is a number, whereas O_1 is an operator. Will lead to probability P_n for creating out of the vacuum a state with n mesons given by $P_n = e^{-|\alpha|}|\alpha|^n/n!$, the Poisson distribution. You'll work out the details in the HW assignment.

• Compute probabilities by squaring the S-maxtrix amplitudes, but have to be careful with the delta functions, since squaring the delta functions would give nonsense.

Warmup: consider quantum mechanics, with $U(t) = Te^{-i\int^t H(t)dt}$,

$$\langle f|U(t)|i\rangle \approx -i\langle f|H_{int}|i\rangle \int_0^t dt e^{i\omega t},$$

where $\omega = E_f - E_i$. If we take $t \to \infty$ first, we get $\delta(\omega)$ and squaring would give nonsense. That's because we're asking the wrong question if we ask about probability for a transition over all time – instead, we should ask about the rate. So keep t finite for now.

[2] This is clear on dimensional grounds, since $[g] \sim m$. More generally, write $a(p) = \sqrt{2E}\widehat{a}(p)$ and $\mathcal{A} = \prod_i \sqrt{2E_i} \prod_f \sqrt{2E_f}\widehat{\mathcal{A}}$.

Squaring gives $P(t) = 2|\langle f|H_{int}|i\rangle|^2(1-\cos\omega t)/\omega^2$. For $t \to \infty$, multiply by $dE_f\rho(E_f)$ and replace $(1-\cos\omega t)/\omega^2 = 4\sin^2(\frac{1}{2}\omega t)/\omega^2 \to \pi t\delta(\omega)$ (using $\int_{-\infty}^{\infty} dx x^{-2}\sin^2 x = \pi$ (hint: $\sin^2 x/x^2 = (2 - e^{i2x} - e^{-i2x})/4x^2$ and close the contour in the correct directions)) to get

$$\dot{P}_{i\to f} = 2\pi|\langle f|H_{int}|i\rangle|^2\rho(E).$$

This is "Fermi's Golden Rule" – it was actually derived by Dirac, but Fermi used it a lot and called it the golden rule. Another aside: Fermi and Dirac independently discovered that spin 1/2 objects must anticommute, and Dirac generously named such objects "Fermions".

Naively taking $t \to \infty$ initially would have given amplitude $\sim \delta(\omega)$, and squaring that would give $\delta(\omega)^2$, which needs to be replaced with $\delta(\omega)2\pi T$, and then divide by T to get the rate. Similarly in field theory, $\delta(p)^2$ should be replaced with probability $\sim \delta(p)$ times phase space volume factors.