10/14 Lecture outline

• The principle of least action. Particle goes from (t_i, x_i) to (t_f, x_f) along some path $x(t)$. How to determine $x(t)$? Two options:

(i) Locally, using $\vec{F} = m\vec{a}$.

(ii) The principle of least action. Among all possible paths, the correct one is that which locally minimizes (stationarizes)

$$S[x(t)] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)), \qquad L = T - U.$$

This is an example of a *functional*. A function like $x(t)$ has input a number, t, and output a number, $x(t)$. For a functional, the input is a function like $x(t)$, and the output is a number, $S[x(t)]$. The number depends on the whole path.

• Example: take $T = \frac{1}{2}m\dot{x}^2$ and $U(x) = mgx$, and $(t_i, x_i) = (0, 0)$ and (t_f, x_f). We know from $F = ma$ that

$$x(t) = -\tfrac{1}{2}gt^2 + v_0 t + x_0,$$

where $x_0 = 0$ and v_0 is determined by $x_f = -\frac{1}{2}gt_f^2 + v_0 t_f$. Plot it; it's a parabola. Think about how it minimizes $S[x(t)]$.

• Other minimized functional examples.

E.g. surface of a soap bubble, minimizes (more precisely stationarizes) the total area.

Fermat's principle: among all possible paths, light takes the one of least (more precisely, stationary) time:

$$T = \int dt = \int \frac{ds}{v} = \frac{1}{c}\int n(x, y) ds, \qquad ds = \sqrt{dx^2 + dy^2},$$

Brachistochrone: what path from $1 \to 2$ such that a sliding object will get there in shortest time?

$$T = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \int_0^{y_f} \sqrt{\frac{x'(y)^2 + 1}{y}} dy.$$

(Answer: a cycloid: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

• Here's how to do these kinds of problems. Consider generally

$$F[y(x)] = \int_{x_1}^{x_2} dx L[y(x), y'(x)].$$

Now vary $y(x) \to y(x) + \delta y(x)$ and we get

$$\delta F = F[f(x) + \delta f(x)] - F[f(x)] = \int_{x_1}^{x_2} dx \left(\frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial y'} \frac{d}{dx} \delta y(x) \right)$$
$$= \int_{x_1}^{x_2} dx \left(\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right) \delta y(x),$$

where we integrated by parts and dropped the boundary term (because we take δy to preserve the endpoint boundary conditions. Write this as

$$\frac{\delta F}{\delta y(x)} = \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'}.$$

The condition that the functional is stationary is that $\frac{\delta F}{\delta y(x)} = 0$:

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0.$$

This is the Euler-Lagrange equation.

- Check brachistochrone $0 = \frac{\delta}{\delta x(y)} T[x(y), x'(y)]$:

$$\frac{\partial f}{\partial x} = \frac{d}{dy} \frac{\partial f}{\partial x'}, \qquad f = \sqrt{x'^2 + 1}/\sqrt{y},$$

Implies $x' = \sqrt{y/(2a - y)}$, yields cycloid above.

- For our problem of interest, $S = \int dtL$ with $L = T - V$, the Euler-Lagrange equation is

$$\frac{\delta S}{\delta x(t)} = \frac{\partial L}{\partial x(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)} = 0.$$