11/30 Lecture outline

• New topic: Hamilton's formulation of mechanics. Newton's in 1687, Lagrange's in 1788, Hamilton's in 1834.

• Generalized coordinates $q_i$. In the Lagrangian description we study $\mathcal{L}(q_i, \dot{q}_i, t)$. Note that it depends on coordinates and their time derivatives, the velocities. The E.L. equations yield Newton's laws, which are second order differential equations for the $q_i$. Hamilton's innovation is to replace them with (coupled) first order differential equations for the coordinates $(q_i, p_i)$. In the Lagrangian description, $p_i = \partial\mathcal{L}/\partial\dot{q}_i$ is not an independent, basic variable. Conversely, in the Hamiltonian description $\dot{q}$ is not an independent, basic variable.

• The Hamiltonian is

$$\mathcal{H}(q_i, p_i, t) = \sum_i p_i \dot{q}_i - \mathcal{L}.$$

Note that the Hamiltonian is written as a function of $p_i$ (and $q_i$), rather than $\dot{q}_i$. How? This is an example of a Legendre transform. Show how this works for the simple case of a single coordinate $q$: $\mathcal{H} = p\dot{q} - \mathcal{L}(q, \dot{q}, t)$. Now compute

$$\begin{aligned} d\mathcal{H} &= dp\dot{q} + pd\dot{q} - \frac{\partial\mathcal{L}}{\partial q}dq - \frac{\partial\mathcal{L}}{\partial\dot{q}}d\dot{q} - \frac{\partial\mathcal{L}}{\partial t}dt \\ &= dp\dot{q} - \frac{\partial\mathcal{L}}{\partial q}dq - \frac{\partial\mathcal{L}}{\partial t}dt, \end{aligned}$$

where on the 2nd line we used $p = \partial\mathcal{L}/\partial\dot{q}$. It follows from the $d(\cdot)$ terms on the RHS that $\mathcal{H}$ is naturally written as a function of $(q, p, t)$. Moreover:

$$\frac{\partial\mathcal{H}}{\partial p}\Big|_{q,t} = \dot{q}, \qquad \frac{\partial\mathcal{H}}{\partial q}\Big|_{p,t} = -\dot{p}, \qquad \frac{\partial\mathcal{H}}{\partial t}\Big|_{p,q} = -\frac{\partial\mathcal{L}}{\partial t},$$

where we used the above expression for $d\mathcal{H}$, and the E.L. equation in the 2nd equality. The first two equalities are called Hamilton's equations. Note also that the total time derivative is

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial p}\dot{p} + \frac{\partial\mathcal{H}}{\partial q}\dot{q} + \frac{\partial\mathcal{H}}{\partial t} = \frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t},$$

where the second equality used Hamilton's equations. So the Hamiltonian is conserved – a constant of the motion – if the it does not depend explicitly on time. That is the case if the Lagrangian does not depend explicitly on time (as we have already before discussed).

The $(q_i, p_i)$ variables are called **phase space.**

• Hamilton's equations:

$$\frac{\partial \mathcal{H}}{\partial p}\Big|_{q,t} = \dot{q}, \qquad \frac{\partial \mathcal{H}}{\partial q}\Big|_{p,t} = -\dot{p},$$

are two first order equations, of the general form $\dot{q} = f(p,q)$ and $\dot{p} = g(p,q)$. (We omit $t$ here, supposing that there is no explicit $t$ dependence). Such DEQs are well-studied by mathematicians. Picture them as giving the dynamics of flow on the $(q,p)$ phase space. Each point $(p,q)$ has a unique orbit of these equations through phase space.

• Harmonic oscillator example:

$$\mathcal{H} = \frac{p^2}{2m} + \tfrac{1}{2}kx^2.$$

Energy conservation gives $\mathcal{H} = E = \frac{1}{2}kA^2$, where $A$ is the maximum amplitude. The phase space orbit $(q,p)$ associated with a given energy $E$ is thus an ellipse, with semi-major axes $A$ in the $q$ direction and $\sqrt{km}A$ in the $p$ direction. The Hamilton equations are

$$\dot{x} = p/m, \qquad \dot{p} = -kx,$$

which gives how a point on the ellipse cycles around the ellipse in time, completing a full circuit in the period $\tau = 2\pi/\omega$, where $\omega = \sqrt{k/m}$. Indeed, the Hamilton equations can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

which is reminiscent of the equations from the previous two lectures, but now they are two first order equations for one degree of freedom, rather than $n$ second order equations for $n$ degrees of freedom. The solutions are anyway found similarly, and we obtain oscillations with frequency $\omega = \sqrt{k/m}$.

• With $n$ degrees of freedom, Hamilton's equations become:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}.$$

These are $2n$ first order equations on the $2n$ dimensional phase space $(q_i, p_i)$. E.g. for a point particle moving in 3 space dimensions, these are equations for orbits in a 6d phase space.

• Liouville's theorem: the $2n$ dimensional velocity vector on phase space, $(\dot{q}_i, \dot{p}_i)$. has vanishing divergence. Consider e.g. the case for $n = 1$ degree of freedom, with $\mathbf{v} = (\dot{q}, \dot{p})$:

$$\nabla \cdot \mathbf{v} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = 0,$$

where the last equality uses the Hamilton equations. It follows from this that the phase space motion preserves volumes in phase space. This is important for the foundation of statistical mechanics.

• Poisson brackets. For any functions $f(q_i, p_i, t)$ and $g(qi, p_i, t)$, define

$$\{f, g\} \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

For example:

$$\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0.$$

It follows from the definition that

$$\{f, g\} = -\{g, f\}, \quad \{f, c\} = 0, \quad \{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$$

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}.$$

Then Hamilton's equations can be written as

$$\dot{q}_i = \{H, q_i\}, \qquad \dot{p}_i = \{H, p_i\}$$

and more generally it follows from Hamilton's equation that, for any function $f(q_i, p_i, t)$ of phase space and time,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}.$$

If $f$ is a conserved quantity, the RHS vanishes. Assuming that it doesn't depend explicitly on $t$, the conserved quantities thus satisfy $\{H, f\} = 0$.