11/21 Lecture outline

• Continue with coupled oscillators. Recall

$$\mathbf{M}\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

As usual, we can solve the equations by taking $\mathbf{x} = Re\mathbf{z}$, with $\mathbf{z} = \mathbf{a}e^{-i\omega t}$. Plugging in, we find that

$$(\mathbf{K} - \omega^2 M)\mathbf{a} = 0,$$

and since we want to have $\mathbf{a} \neq 0$, the determinant of the matrix multiplying it must vanish. For n oscillators this gives n solutions ω_i and $\mathbf{a}_{i=1\ldots n}$. We were considering the example

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - \tfrac{1}{2}k_1x_1^2 - \tfrac{1}{2}k_2(x_1 - x_2)^2 - \tfrac{1}{2}k_3x_2^2$$

Taking $k_1 = k_3$ and $k_2 = k_{int}$ and $m_1 = m_2$, get $\omega_1 = \omega_- = \sqrt{\frac{k_1}{m}}$ and $\omega_2 = \omega_+ = \sqrt{\frac{k_1+2k_{int}}{m}}$, with

$$\mathbf{a}_{(\mathbf{1})} = \begin{pmatrix} A \\ A \end{pmatrix}, \qquad \mathbf{a}_{(\mathbf{2})} = \begin{pmatrix} A \\ -A \end{pmatrix}$$

Example: $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. Then $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$.

If we define normal coordinates by If we define normal coordinates by $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$, then the two normal modes are

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} = \begin{pmatrix} A_1\cos(\omega_1 t - \delta_1) \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ A_2\cos(\omega_2 t - \delta_2) \end{pmatrix},$$

and the general solution is a general superposition of them.

• For weakly coupled oscillators, $k_{int} \ll k_1$, and then $\omega_2 \approx \omega_1 + \epsilon$, where $\epsilon \equiv \frac{k_{int}}{k_1}$. Find interesting solutions like

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A\cos\epsilon t\cos\omega_0 t \\ A\sin\epsilon t\sin\omega_0 t \end{pmatrix}.$$

• General case, where there are n coupled degrees of freedom, with coordinates q_i, for $i = 1\ldots n$. Suppose that there is a stable equilibrium when all $q_i = 0$. For small displacements from equilibrium, we get in effect n coupled harmonic oscillators, with equations like above

$$\mathbf{M}\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{K}\mathbf{x}, \tag{1}$$

$$\mathbf{x} = Re\mathbf{a}e^{-i\omega t}, \tag{2}$$

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = 0. \tag{3}$$

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0. \tag{4}$$

To see this expand the general kinetic and potential energies to quadratic order in the small displacements q_i:

$$T = \tfrac{1}{2}\sum_{ij} A_{ij}(q)\dot{q}_i\dot{q}_j \approx \tfrac{1}{2}\sum_{ij} M_{ij} q_i q_j,$$

where $M_{ij} = A_{ij}(q_i = 0)$ and

$$U(q) \approx \tfrac{1}{2}\sum_{ij} K_{ij} q_i q_j,$$

where $K_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j}|_{q_i=0}$ are the coefficients in our Taylor's expansion of U to quadratic order in small fluctuations q_i. The equation of motion can be written in matrix notation exactly as in (1). The solution again is of the form (2). Again we find the normal modes from (3) and (4), which now has n normal-mode solutions. The general solution is again a superposition of the n normal modes, which can be written in normal mode coordinates as

$$\mathbf{x} = \sum_{j=1}^{n} \xi_j(t)\mathbf{a}_{(j)}.$$

As before, each $\xi_j(t)$ behaves as a decoupled harmonic oscillator coordinate, with frequency ω_j, so

$$\xi_j(t) = A_j \cos(\omega_j t + \varphi_j).$$

The 2n constants A_j and φ_j are constants of integration, determined e.g. by the n initial positions and velocities.

- An example: double pendulum:

$$T = \tfrac{1}{2} m_1 L_1^2 \dot{\phi}_1^2 + \tfrac{1}{2} m_2 (L_1^2 \dot{\phi}_1^2 + 2L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + L_2^2 \dot{\phi}_2^2).$$

(Discuss the faster but perhaps tricky, and the slower but straightforwardly reliable ways to get the m_2 kinetic energy.)

$$U = m_1 g L_1(-\cos\phi_1) + m_2 g(-L_1\cos\phi_1 - L_2\cos\phi_2).$$

For small oscillations we expand $\sin\phi_i \approx \phi_i$ and $\cos\phi_i \approx 1 - \frac{1}{2}\phi_i^2$. Keeping terms only of order ϕ^2 and lower, the E.L. equations for small oscillations yield

$$\mathbf{M}\frac{d^2}{dt^2}\phi = -\mathbf{K}\phi,$$

with

$$\mathbf{M} = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix}, \qquad K = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}.$$

As before, we solve this via $\phi(t) = Re(\mathbf{A}e^{-i\omega t})$ where $\mathbf{A}$ and the ω are determined by solving

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0.$$

Let's write it in gory detail for the case of $m_1 = m_2 = m$ and $L_1 = L_2 = L$, with $\sqrt{g/L} \equiv \omega_0$:

$$mL^2 \begin{pmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0,$$

from which it follows that the two normal mode frequencies are

$$\omega_1^2 = (2-\sqrt{2})\omega_0^2, \qquad \omega_2^2 = (2+\sqrt{2})\omega_0^2,$$

with corresponding solutions

$$\mathbf{a}_{(1)} = A_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \mathbf{a}_{(\mathbf{2})} = A_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

• Example of three coupled pendulums. $T = \sum_{i=1}^{3} \frac{1}{2}mL^2\dot{\phi}_i^2$, $U_{grav} \approx \frac{1}{2}mgL\sum_i \phi_i^2$, and $U_{spring} \approx \frac{1}{2}kL^2((\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2)$. Set $m = L = 1$, since we can restore them later. So $\mathbf{M} = m\mathbf{1}$ and

$$\mathbf{K} = \begin{pmatrix} g+k & -k & 0 \\ -k & g+2k & -k \\ 0 & -k & g+k \end{pmatrix},$$

Then the characteristic equation gives $\omega_1^2 = g$, $\omega_2^2 = g+k$, $\omega_3^2 = g+3k$, with modes

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$