11/18 Lecture outline

• New topic: coupled oscillators and normal modes. Chapter 11 in Taylor. Example from book of two carts and 3 coupling springs:

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - \tfrac{1}{2}k_1x_1^2 - \tfrac{1}{2}k_2(x_1 - x_2)^2 - \tfrac{1}{2}k_3x_2^2.$$

The Euler-Lagrange equations can be written in matrix notation as

$$\mathbf{M}\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

where $\mathbf{x}$ is a column vector of the two positions and

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}.$$

As with the simple harmonic oscillator, the equations of motion are linear in $\mathbf{x}$, so we have superposition. As usual, we can solve the equations by taking $\mathbf{x} = Re\mathbf{z}$, with $\mathbf{z} = \mathbf{a}e^{-i\omega t}$. Plugging in, we find that

$$(\mathbf{K} - \omega^2 M)\mathbf{a} = 0,$$

and since we want to have $\mathbf{a} \neq 0$, the determinant of the matrix multiplying it must vanish. For simplicity, consider the case $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. Then we have

$$\det(\mathbf{K} - \omega^2 M) = \det\begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} = (k - m\omega^2)(3k - m\omega^2),$$

and the condition that this vanishes determines the two normal modes of oscillation: $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{3k}{m}}$. We now solve for the $\mathbf{a}$ such that $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ for each of the two normal modes. Doing so, we find that the two normal modes have

$$\mathbf{a}_{(\mathbf{1})} = \begin{pmatrix} A \\ A \end{pmatrix}, \qquad \mathbf{a}_{(\mathbf{2})} = \begin{pmatrix} A \\ -A \end{pmatrix}$$

respectively, where A is an arbitrary value for the amplitude. If we define normal coordinates by $\xi_1 = \frac{1}{2}(x_1 + x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$, then the two normal modes are

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} = \begin{pmatrix} A_1\cos(\omega_1 t - \delta_1) \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ A_2\cos(\omega_2 t - \delta_2) \end{pmatrix},$$

and the general solution is a general superposition of them.

• An interesting special case: two weakly coupled oscillators. When $k_1 = k_3 = k$, we obtain the normal modes $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{k+2k_2}{m}}$. For weakly coupled oscillators, $k_2 \ll k$, and then $\omega_2 \approx \omega_1 + \frac{k_2}{k}$. We then have $\omega_0 \equiv \frac{1}{2}(\omega_1 + \omega_2) \approx \sqrt{\frac{k}{m}}$ and $\frac{1}{2}(\omega_2 - \omega_1) = \frac{k_2}{k} \equiv \epsilon$. Find interesting solutions like

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A\cos\epsilon t\cos\omega_0 t \\ A\sin\epsilon t\sin\omega_0 t \end{pmatrix}.$$