11/15 Lecture outline

• Last times: Two body central force problems, reduces to solving the 1d problem:

$$\mu\frac{d^2r}{dt^2} = -\frac{dU_{eff}(r)}{dr}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}, \qquad \ell = \mu r^2\dot{\phi},$$

$$H = \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = E = \text{constant}.$$

Eliminate t and solve for $r(\phi)$. Easier to change variables to $u = 1/r$, get

$$u''(\phi) + u + \frac{\mu}{\ell^2 u^2}F(r) = 0.$$

Kepler orbits: $U(r) = -k/r$, so $F(r) = -k/r^2$. (Sign is chosen so that $k > 0$ corresponds to an attractive force). Get

$$u''(\phi) = -u(\phi) + k\mu/\ell^2,$$

which is like the free particle, if we substitute $w = u - k\mu/\ell^2$, so

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}, \qquad c \equiv \frac{\ell^2}{k\mu}. \tag{1}$$

where ϵ is a constant, which can be written in terms of the energy as

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

Indeed,

$$E = U_{eff}(r_{min}) = -\frac{k}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{k^2\mu}{2\ell^2}(\epsilon^2 - 1).$$

So $\epsilon < 1$ gives bounded orbits, and $\epsilon > 1$ gives unbounded orbits. For $\epsilon < 1$ the equation is an ellipse (with special case being a circle for $\epsilon = 0$). For $\epsilon > 1$ it is a hyperbola. For $\epsilon = 1$ it is a parabola.

• For $E < 0$ (bound orbits) get $\epsilon < 1$, and the above conic section is an ellipse. The ellipse has major and minor semi-axes given by

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a = \frac{c}{1-\epsilon^2}, \; b = \frac{c}{\sqrt{1-\epsilon^2}}, \; d = a\epsilon.$$

So

$$\frac{b}{a} = \sqrt{1-\epsilon^2}, \qquad r_{min} = \frac{c}{1+\epsilon}, \qquad r_{max} = \frac{c}{1-\epsilon}.$$

Also,

$$r_{min} = \frac{c}{1+\epsilon} = a(1-\epsilon), \quad r_{max} = \frac{c}{1-\epsilon} = a(1+\epsilon).$$

Since $1-\epsilon^2 = -2E\ell^2/\mu k^2 = 2|E|\ell^2/\mu k^2$ we have

$$a = \frac{k}{2|E|}, \qquad b = \frac{\ell}{\sqrt{2\mu|E|}}.$$

The period of revolution is given by recalling $dA/dt = \ell/2\mu$ (Kepler's 2nd law), so the period is $\tau = A/\dot{A} = 2\pi ab\mu/\ell$ so

$$\tau = 2\pi a^{3/2}\sqrt{\frac{\mu}{k}} = \pi k\sqrt{\frac{\mu}{2|E|^3}}.$$

Note that the period is uniquely determined by the energy.

For a comet or planet orbiting the sun, $k = Gm_1m_2 \approx G\mu M_s$ so $\tau^2 \approx 4\pi^2a^3/GM_s$; Kepler's 3rd law. History lesson: Tycho Brahe (1546-1601) took the data, and hired Kepler in 1600 as his assistant to help and to interpret the data. The planet's average distance relative to the earth-sun distance were obtained, and that gave $\tau^2 \sim a^3$. The actual earth-sun distance was obtained (to 7% error) in 1672 by Giovanni Cassini by using parallax (to obtain the earth-mars distance). In 1672, this was used to determine the constancy of the speed of light by astronomer Olaf Roemer.

• For $\epsilon = 1$, get $y^2 = c^2 - 2cx$, a parabola. For $\epsilon > 1$, get hyperbola:

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1.$$

• Mechanical similarity. Suppose that all lengths are rescaled: $\vec{r}_i \to \alpha\vec{r}_i$. Suppose that the potential energy is homogenous function of degree n, i.e. $U(\alpha\vec{r}_i) = \alpha^n U(\vec{r}_i)$. Examples: for $U = k/r$, we have $n = -1$; for $U = \frac{1}{2}kr^2$ we have $n = 2$. Suppose that we also scale time as $t \to \beta t$. Then velocities scale as $\vec{v} \to \frac{\alpha}{\beta}\vec{v}$, and kinetic energy scales as $T \to \frac{\alpha^2}{\beta^2}T$. Assuming that the potential scales homogeneously, the lagrangian also scales homogeneously if we take $\alpha^2/\beta^2 = \alpha^n$, i.e. $\beta = \alpha^{1-\frac{1}{2}n}$. Since the scale is just an overall factor, the equations of motion are unchanged. This is interesting: it implies that homogeneous potentials have similar solutions, differing only by rescalings, with properties simply related. Let a be a length scale in a solution, and a' be a length scale in the rescaled solution, with $a'/a = \alpha$. We then have

$$\frac{t'}{t} = \alpha^{1-\frac{1}{2}n}, \ \frac{v'}{v} = \alpha^{\frac{1}{2}n}, \ \frac{E'}{E} = \alpha^n, \ \vec{L}' = \vec{L}\alpha^{1+\frac{1}{2}n}.$$

For example, for $U = -k/r$, $n = -1$, and we immediately obtain that the period scales with the orbit size as $\tau \sim a^{1-\frac{1}{2}n} = a^{3/2}$.

Recall also the viral theorem:

$$2\langle T\rangle = n\langle U\rangle.$$

Taking $E = T + U$ a constant, we have $E = \langle T\rangle + \langle U\rangle = (1 + \frac{1}{2}n)\langle U\rangle = (1 + \frac{2}{n})\langle T\rangle$.

• Orbit change by tangential thrust at prigee. Initial and final orbits have the same perigee

$$r_{min} = \frac{c_1}{1+\epsilon_1} = \frac{c_2}{1+\epsilon_2}.$$

The velocity at perigee changes to $v_2 = \lambda v_1$. (The two orbits are not trivially related by mechanical similarity, since not all lengths are related by the same rescaling.) Since $\ell_2 = \lambda \ell_1$, we have $c_2 = \lambda^2 c_1$ and thus $\epsilon_2 = \lambda^2\epsilon_1 + \lambda^2 - 1 > \epsilon_1$, i.e. the orbit becomes more eccentric for $\lambda > 1$.

• Next and final topic: coupled oscillators and normal modes.

Example (from book) of two carts and 3 coupling springs:

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - \tfrac{1}{2}k_1x_1^2 - \tfrac{1}{2}k_2(x_1 - x_2)^2 - \tfrac{1}{2}k_3x_2^2.$$

The Euler-Lagnrange equations can be written in matrix notation as

$$\mathbf{M}\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

where $\mathbf{x}$ is a column vector of the two positions and

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}.$$