11/14 Lecture outline

• Last times: Two body central force problems, reduces to solving the 1d problem:

$$\mu \frac{d^2 r}{dt^2} = -\frac{dU_{eff}(r)}{dr}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}, \qquad \ell = \mu r^2 \dot{\phi},$$

$$H = \tfrac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = E = \text{constant}.$$

Solution for $r(t)$ and $\phi(t)$ from

$$t = \int_{r_0}^{r(t)} \frac{dr}{\sqrt{\frac{2}{\mu}(E - U(r) - \frac{\ell^2}{2\mu r^2})}},$$

$$\phi - \phi_0 = \ell \int_0^t \frac{dt}{\mu r^2(t)}.$$

• Orbit equations have solution $r = r(t)$ and $\phi = \phi(t)$. Let's study the shape of the trajectory rather than the t dependence. Eliminating the parameter t, we can solve for $r = r(\phi)$. To do this, use

$$\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi},$$

where $u = 1/r$ is introduced for convenience. So

$$\frac{dr}{dt} = -\frac{\ell}{\mu}\frac{du}{d\phi}, \qquad \frac{d^2 r}{dt^2} = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2 u}{d\phi^2},$$

and the r EOM becomes (with $F(r) = -dU/dr$)

$$u''(\phi) + u + \frac{\mu}{\ell^2 u^2} F(r) = 0.$$

• For circular orbits, $u = u_0$ =constant. For nearly circular orbits, we can write $u = u_0 + \delta(\phi)$ and expand the above to find an equation for $\delta(\phi)$. Let's instead write it in terms of the original variable r, so $r = r_0 + \eta(\phi)$ and then plug into the equation above to find

$$\frac{d^2\eta}{d\phi^2} = -\beta^2 \eta,$$

where

$$\beta^2 \equiv 3 - \frac{\mu r_0^4}{\ell^2} F'(r_0).$$

A solution is $\eta(\phi) = \eta_0 \cos\beta\phi$. The maximum is chosen at $\phi_n = 2\pi n/\beta$.

• Consider the general orbit equation above for the example of a free particle, $F(r) = 0$. The solution is $u(\phi) = r_0^{-1}\cos(\phi - \delta)$, the equation of a straight line. Good.

• Kepler orbits: $U(r) = -k/r$, so $F(r) = -k/r^2$. (Sign is chosen so that $k > 0$ corresponds to an attractive force). Get

$$u''(\phi) = -u(\phi) + k\mu/\ell^2,$$

which is like the free particle, if we substitute $w = u - k\mu/\ell^2$, so

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}, \qquad c \equiv \frac{\ell^2}{k\mu}. \tag{1}$$

where ϵ is a constant, which can be written in terms of the energy as

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

• Another option is to solve for $r(\phi)$ by using energy conservation at the outset. As usual, this is better because $F = ma$ gives a 2nd order differential equation, whereas energy conservation does one of those integrals for us, leaving just a first order equation remaining to integrate. Using the relation (see above) $\frac{d}{dt} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$, we get

$$\frac{dr}{dt} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi}$$

and substituting into energy conservation then gives

$$E = \tfrac{1}{2}\mu\left(\frac{\ell}{\mu r^2}\frac{dr}{d\phi}\right)^2 + U_{eff}(r),$$

which we can use to solve for $dr/d\phi$, and then integrate the equation to obtain

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{\ell dr/r^2}{\sqrt{2\mu(E - U_{eff}(r))}}$$

For the particular case of $U = -k/r$ the integral indeed leads to the $r(\phi)$ given above.

• Continue with $U = -k/r$. The energy is

$$E = U_{eff}(r_{min}) = -\frac{k}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{k^2\mu}{2\ell^2}(\epsilon^2 - 1).$$