11/4 Lecture outline

• Next topic: Two body central force problems. Conservative force $\vec{F}_{12} = -\frac{\partial}{\partial \vec{r}_1} U(\vec{r}_1, \vec{r}_2)$. Translational invariance: $U(\vec{r}_1, \vec{r}_2) = U(\vec{r}_1 - \vec{r}_2)$ (implies conservation of total momentum). Central (rotational invariance): $U(\vec{r}_1 - \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$ (implies conservation of angular momentum). Introduce $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and $r = |\vec{r}|$, and $U = U(r)$.

• Examples:

$$U(r) = -\frac{Gm_1 m_2}{r} \qquad U(r) = C\frac{q_1 q_2}{r},$$

for gravitational and electric forces (with $C = 1/4\pi\epsilon_0$ in MKS units). These both have $U(r) \sim 1/r$, which gives an inverse square force law. This case has a number of special properties: inverse square force is related to a conserved *flux*, e.g. the flux of electric field, related by Gauss' law to the total charge inside. Also, we'll discuss a special conserved vector quantity for $1/r$ potentials. But let's now consider general $U(r)$.

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$$\mathcal{L} = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r).$$

• CM and relative coordinates:

$$\vec{R} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad M \equiv m_1 + m_2.$$

Useful because

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = M\dot{\vec{R}} = \text{constant}.$$

Convert using

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r},$$

to obtain

$$\mathcal{L} = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2}\mu \dot{\vec{r}}^2 - U(\vec{r}),$$

where the "reduced mass" is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \text{i.e.} \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

The problem has separated into the CM and the relative motion. Can even go to the CM inertial frame, in which $\dot{\vec{R}} = 0$, and simply study the relative motion. The relative motion is that of a particle of mass μ, in the central force potential $U(r)$.

• $\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$. In the CM frame, $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$. Conservation of angular momentum implies then that $\vec{r} \times \dot{\vec{r}}$ is a constant, which implies that $\vec{r}$ and $\dot{\vec{r}}$ lie in an unchanging plane.

- So the problem reduces to a single particle moving in 2d plane:

$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r),$$

so

$$p_\phi = \mu r^2\dot{\phi} = \ell = \text{constant} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{\mathrm{r}}} = \frac{\partial\mathcal{L}}{\partial\mathrm{r}},$$

which gives

$$\mu\frac{d^2}{dt^2}r = -\mu r\dot{\phi}^2 - \frac{dU}{dr}$$

and since $F_{cf} = \mu r\dot{\phi}^2 = \ell^2/\mu r^3$ we can reduce the motion to solving the 1d problem:

$$\mu\frac{d^2r}{dt^2} = -\frac{dU_{eff}(r)}{dr}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}.$$

(Note that we substituted $\dot{\phi} = \ell/\mu r^2$ only *after* computing the r equations of motion, and then wrote U_{eff}. Eliminating $\dot{\phi}$ too soon gives a wrong sign for U_{eff}.)