11/2 Lecture outline

• Example of hoop of radius a rolling on incline plane, without slipping. Define $I = ma^2$ (the equations then apply more generally, for an object of moment of inertia I).

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2 + mgx\sin\phi,$$

where x is distance down from the top, and ϕ is the incline angle, and θ is the coordinate of the hoop. The no slip condition is $x - a\theta =$constant. Get

$$m\frac{d^2x}{dt^2} = mg\sin\phi - \lambda, \qquad I\frac{d^2\theta}{dt^2} = \lambda a, \qquad a\dot{\theta} = \dot{x}.$$

Using $I = ma^2$, get

$$\frac{d^2x}{dt^2} = \tfrac{1}{2}g\sin\phi, \qquad \frac{d^2\theta}{dt^2} = g\sin\phi/2a, \qquad \lambda = F_{constr} = \tfrac{1}{2}Mg\sin\phi.$$

• If $L = T(\dot{q}_i) - U(q_i)$ with T purely quadratic in velocities, then

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T = \sum_i p_i \dot{q}_i.$$

In this case we get $H = \sum_i p_i\dot{q}_i - L = T + U$. In this case we also have the *Viral Theorem*, which follows from the above, written as

$$2T = \frac{d}{dt}\left(\sum_i p_i q_i\right) - \sum_i q_i \dot{p}_i,$$

and time averaging gives gives

$$\langle 2T \rangle = \sum_i q_i \frac{\partial U}{\partial q_i} = k\langle U \rangle,$$

where in the last equality we assumed that U is homogeneous of degree k in the coordinates, $U \sim q^k$. Using $T + U = E$ time independent, we get

$$\langle U \rangle = 2E/(k+2), \qquad \langle T \rangle = kE/(k+2).$$

Example: harmonic oscillator has $k = 2$. Inverse-square forces have $k = -1$.