10/31 Lecture outline

• Last time, discuss systems with constraints. Example: pendulum:

$$\mathcal{L} = \tfrac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell\cos\phi$$

has 1 d.o.f., namely ϕ. Alternatively, we could use $x_{bob} = \ell\sin\phi$ and $y_{bob} = -\ell\cos\phi$, with the constraint $x_{bob}^2 + y_{bob}^2 = \ell^2$. This is an example of a *holonomic constraint*, which more generally are constraints of the form:

$$f(q_i, t) = 0.$$

We want to extremize the action, subject to the requirement that the variations δq_i should satisfy the constraint. A way to do this is to introduce a Lagrange multiplier. We replace the Lagrangian with

$$\mathcal{L} + \lambda f,$$

where λ is the Lagrange multiplier.

We treat the q_i as if they were all independent, and treat λ as another independent coordinate. The E.L. equations then become:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i} + \lambda\frac{\partial f}{\partial q_i},$$

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\lambda}} = \frac{\partial\mathcal{L}}{\partial\lambda} = f(q_i).$$

The lagrange multiplier terms can be interpreted as the **constraint** force: $F_{constr,i} = \lambda\frac{\partial f}{\partial q_i}$.

• Note that constraint forces don't do work: $dW_{constr} = F_{constr,i}dq_i = \lambda\frac{\partial f}{\partial q_i}dq_i = \lambda df = 0$, since we hold $f = 0$, so $df = 0$.

• Let's apply this to the above example:

$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(x^2 + y^2 - \ell^2).$$

we then get

$$m\frac{d^2x}{dt^2} = 2\lambda x, \qquad m\frac{d^2y}{dt^2} = -mg + 2\lambda y,$$

$$x^2 + y^2 = \ell^2.$$

So $F_{constr,i} = (2\lambda x, 2\lambda y)$. We can use this to solve for the tension in the pendulum string.

We can alternatively work in polar coordinates, and keep r as a variable, and then use a Lagrange multiplier to set $r = \ell$. We then have:

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mg\ell\cos\phi + \lambda(r - \ell).$$

- Atwood machine example:

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2,$$

with the constraint $x_1 + x_2 = \ell$ a constant. One way to solve the problem is to just eliminate x_2 by writing $x_2 = \ell - x_1$. This is fine, unless we want to solve for the tension in the string. In that case, we keep both x_1 and x_2 and impose the constraint with a Lagrange multiplier. This gives

$$m_1\frac{d^2x_1}{dt^2} = m_1 g - F_t, \qquad m_2\frac{d^2x_2}{dt^2} = m_2 g - F_t,$$

with $F_t = -\lambda$. Solve as

$$\frac{d^2x_1}{dt^2} = \frac{(m_1 - m_2)g}{m_1 + m_2}, \qquad \frac{d^2x_2}{dt^2} = \frac{(m_2 - m_1)g}{m_1 + m_2}, \qquad F_t = -\lambda = \frac{2m_1 m_2 g}{m_1 + m_2}.$$