10/26 Lecture outline

• Last time, Noether's theorem: continuous symmetries implies conservation laws. If $\mathcal{L}$ is invariant under $q_i \to q_i(\xi)$, with infinitesimal change $\delta q_i = \frac{\partial q_i}{\partial \xi}\delta\xi$, then

$$0 = \delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i),$$

from which it follows that

$$\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i(\xi)}{\partial \xi}|_{\xi=0}$$

is a conserved quantity.

• If L is independent of a coordinate q ("cyclic"), then there is a symmetry $q \to q+\xi$, and the conserved quantity is simply the corresponding conjugate momentum $p = \partial L/\partial \dot{q}$. The above result is more non-trivial when the symmetry is less obvious. Example: particle in 2d with $U = U(r)$ has symmetry $\phi \to \phi + \xi$, gives conserved $\ell = mr^2\dot{\phi}$. Or use above in rectangular coordinates, infinitesimal rotation $\delta x = -d\xi y$, $\delta y = d\xi x$, so $\delta\vec{x} = d\xi\hat{z}\times\vec{x}$, gives conserved $\vec{L}_z$.

• Example of helical symmetry: $U(\rho, \phi, z) = U(\rho, a\phi + z)$ has symmetry $\phi \to \phi + \xi$, $z \to z - \xi a$, gives conserved quantity $m\rho^2\dot{\phi} - ma\dot{z}$.

• Isolated system of particles has translation symmetry $\vec{x}_a \to \vec{x}_a + \xi\vec{n}$. Gives conserved quantity $\vec{n}\cdot\vec{P}$, where $\vec{P}$ is the total momentum. Conservation of total momentum.

• Isolated system has rotation symmetry: $\delta\vec{x}_a = \xi\hat{n}\times\vec{x}_a$. Gives conserved quantity

$$\sum_a \frac{\partial L}{\partial \dot{\vec{x}}_a}\cdot\hat{n}\times\vec{x}_a = \hat{n}\cdot\vec{L}, \qquad \vec{L} = \sum_a \vec{x}_a \times \vec{p}_a,$$

i.e. conservation of total angular momentum.