10/21 Lecture outline

• Last time, $S[q_i, \dot{q}_i, t] = \int dt L(q_i, \dot{q}_i, t)$, extremized for the $q_{i*}(t)$ solving EL equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

We showed that these equations imply that

$$\frac{d}{dt}[\sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L] = -\frac{\partial L}{\partial t},$$

so if L doesn't depend explicitly on t, then

$$\sum_i q_i \frac{\partial L}{\partial \dot{q}_i} - L = constant (= E).$$

Generalized coordinates $q_i(t)$, with generalized momenta $p_i \equiv \frac{\partial L}{\partial \dot{q} i}$, then the Euler-Lagrange equations can be written as

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} p_i,$$

which is a generalized version of $\vec{F} = \frac{d\vec{p}}{dt}$. Note that, if the Lagrangian does not depend explicitly on a coordinate q_i, then the conjugate momentum p_i is conserved.

Continue with our example, pendulum: $L = \frac{1}{2} m\ell^2 \dot{\phi}^2 - mg\ell(1 - \cos\phi)$. Let $q = \phi$ be the generalized coordinate, and then the generalized momentum is $p = m\ell^2\dot{\phi}$, which is just the angular momentum. Euler Lagrange equations give

$$-mg\ell \sin\phi = m\ell^2 \frac{d}{dt}\dot{\phi},$$

which is the expected $\tau = I\alpha$ result from $\vec{\tau} = \frac{d}{dt}\vec{L}$.

• Example: particle in polar coordinates:

$$L = \tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2) - U(\rho, \phi),$$

$$m\rho\dot{\phi}^2 - \frac{\partial U}{\partial \rho} = \frac{d}{dt}(m\dot{\rho}),$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(m\rho^2\dot{\phi}).$$

$$E = \tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2) + U.$$

• Comment on dimensional analysis: $[Lagrangian] \sim [E] \sim ML^2/T^2$. Generalized coordinates might have units of length (like x) or no units (like ϕ) etc. Units work out consistently in any case, as in the above examples.

• Another comment: can always change $L \to L + \frac{d}{dt}G(q, \dot{q}, t)$ without affecting the EL equations of motion. The Lagrangian is inherently ambiguous, in a way that doesn't matter anyway at the end of the day.

• More examples: sliding point mass on sliding wedge. $T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{x})^2 + \frac{1}{2}m\dot{y}^2$, with $y = x\tan\alpha$, and $U = mgx\tan\alpha$.

• Pendulum attached to moving support: support at $x_s(t)$ (given) and mass m bob at $x_{bob} = x_s + \ell\sin\phi$, $y_{bob} = -\ell\cos\phi$. So $L = \frac{1}{2}m(\dot{x}_s^2 + 2\dot{x}_s\ell\cos\phi\dot{\phi} + \ell^2\dot{\phi}^2) + mg\ell\cos\phi$. Compute EL equations for $\phi(t)$, see $d^2x_s/dt^2\cos\phi$ enters as a forcing term.

• Pendulum attached to mass on spring. Coordinates (x, θ), with

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}_{bob}^2 + \dot{y}_{bob}^2) = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta}$$

and $U = \frac{1}{2}kx^2 - mg\ell\cos\theta$, where we used $x_{bob} = x + \ell\sin\theta$, $y_{bob} = -\ell\cos\theta$.

• Note that $T = \frac{1}{2}m(ds^2/dt^2)$, so can use e.g. $ds^2 = dr^2 + r^2d\theta^2$ or $ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ if we want to use polar or spherical coordinates.

• Next time, "cyclic" coordinates ($\partial L/\partial q_{cyclic} = 0$) and conservation laws. E.g. example of motion in 2d central potential $U = U(r)$,

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

so ϕ is a cyclic coordinate, $\partial L/\partial\phi = 0$, and correspondingly $p_\phi = mr^2\dot{\phi} = \ell$ is conserved. This is related to the rotation symmetry, $\phi \to \phi+$ constant, as we'll soon discuss. The r equation of motion (EOM) is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \to m\frac{d^2}{dt^2}r = mr\dot{\phi}^2 - U'(r).$$

We can now eliminate $\dot{\phi}$ in favor of ℓ to get

$$m\frac{d^2r}{dt^2} = \frac{\ell^2}{mr^3} - U'(r) \equiv -\frac{d}{dr}U_{eff}, \qquad U_{eff} = U(r) + \frac{\ell^2}{2mr^2}.$$

Here U_{eff} is an effective potential which accounts for the centrifugal force of the rotating object. The energy is also conserved:

$$H = p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} = \frac{p_r^2}{2m} + U_{eff}(r).$$