10/17Lecture outline

• Last time: consider a general functional $F[y(x)] = \int_{x_1}^{x_2} dx f(y(x), y'(x), x] dx$. The variation is

$$\delta F \equiv F[y + \delta y] - F[y] = \int dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y(x).$$

In the second term we integrated by parts and dropped the boundary term (because we take δy to preserve the endpoint boundary conditions. The functional is therefore stationary for $y(x) = y_*(x)$ which solves the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.$$

Our functional of particular interest is the action, $S[q(t)] = \int_{t_1}^{t_2} dt L[q(t), \dot{q}(t), t]$, where $q(t)$ is some (generalized) coordinate. The Euler-Lagrange equation for least (stationary) action is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0.$$

If there are many coordinates $q_i(t)$, the action should be stationary for all of them, so each one is determined by an independent Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0.$$

• Apply to some examples.

Minimal area: $A[y] = \int_{x_1}^{x_2} dx 2\pi y \sqrt{1 + y'^2} \equiv \int dx L$. Because the integrand doesn't explicitly depend on x, the Euler-Lagrange equations in this case imply that

$$y' \frac{\partial L}{\partial y'} - L = \text{const},$$

implying that $y = b \cosh((x - a)/b)$, where a and b are constants; this is a catenary. (Also shape of drooping stretched rope.)

Fermat principle and similar problems, e.g. Brachistochrone:

$$T = \int_{\theta_1}^{\theta_2} \frac{1}{v(x, y)} \sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2} d\theta,$$

where we introduced the parameter θ as a convenience (alternatively, we can work in terms of $y(x)$ or $x(y)$). Let's call $x(\theta) = q_1(\theta)$ and $y(\theta) = q_2(\theta)$. The stationary path has

$$-\frac{1}{v^2} \sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2} \frac{\partial v}{\partial q_i} - \frac{d}{d\theta} \left(\frac{1}{v \sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2}} \frac{dq_i}{d\theta} \right) = 0.$$

In particular, for the Brachistochrone, $v = \sqrt{2gy}$, we get

$$\frac{d}{d\theta}\left(\frac{\frac{dx}{d\theta}}{\sqrt{y}\sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2}}\right) = 0,$$

$$-\tfrac{1}{2}y^{-3/2}\sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2} - \frac{d}{d\theta}\left(\frac{\frac{dy}{d\theta}}{\sqrt{y}\sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2}}\right) = 0.$$

The solution is a cycloid: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

• Return to our problem of interest, $S = \int dtL$ with $L = T - V$. The action is stationary if

$$\frac{\delta S}{\delta x(t)} = \frac{\partial L}{\partial x(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}(t)} = 0.$$

If we take $L = \frac{1}{2}m\dot{x}^2 - U(x)$, then the above yields

$$-\frac{\partial U}{\partial x} - \frac{d}{dt}m\dot{x} = 0,$$

which is just $F = ma$. Q: is this just a more complicated rewriting of something we already knew from Newton's Principia? A: Basically, yes - but it's useful! We can write T and V independent of any choice of coordinates, and they're scalar rather than vector quantities. Aside: there's a nice way to formulate QM which gives the action S some independent physical meaning.

• Generalized coordinates $q_i(t)$, with generalized momenta $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$, then the Euler-Lagrange equations can be written as

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}p_i,$$

which is a generalized version of $\vec{F} = \frac{d\vec{p}}{dt}$. Note that, if the Lagrangian does not depend explicitly on a coordinate q_i, then the conjugate momentum p_i is conserved. If the Lagrangian does not depend explicitly on time, then

$$E = \sum_i p_i\dot{q}_i - L$$

is conserved.