10/26 Lecture outline

⋆ **Reading: Luke, chapter 7**

• Last time, phase space factors. Put the system in a box of volume V. The momenta are quantized and, as usual, there are $Vd^3\vec{k}/(2\pi)^3$ states with $\vec{k}$ in the range $d^3\vec{k}$. Interested in computing probabilities, $P = |\langle f|i\rangle|^2/\langle f|f\rangle\langle i|i\rangle$. Use e.g. $\langle k|k\rangle = (2\pi)^3 2\omega\delta^3(0)$ and replace $\delta^3(0) \to V$. Put these normalization factors into correct normalization of initial and final states:

$$\langle f|(S-1)|i\rangle_{VT} = i\mathcal{A}_{fi}^{VT}\delta^4(p_F - p_I)\prod_f \frac{1}{\sqrt{2\omega_k V}}\prod_i \frac{1}{\sqrt{2\omega_k V}},$$

where the factors account for the relativistic normalization of the states. Squaring, with the replacement $(2\pi^4\delta^4(p))^2 \to VT(2\pi)^4\delta^4(p)$ (since $\int d^4x e^{i0\cdot x} = VT$) get that the probability per unit time is

$$|\mathcal{A}_{fi}|^2 VD\prod_i \frac{1}{2E_iV},$$

where

$$D = (2\pi)^4\delta^4(p_F - p_I)\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}.$$

Decays: differential decay probability per unit time: $d\Gamma = \frac{1}{2M}|\mathcal{A}_{fi}|^2 D$. Integrate over all possible final states to get $\Gamma = 1/\tau$ where τ is the lifetime.

Cross sections: the number of scatterings per unit time is $dN = Fd\sigma$, where F is the flux. So

$$d\sigma = \frac{\mathcal{A}_{fi}^2}{4E_1E_2V}\ D\ \frac{V}{|\vec{v}_1 - \vec{v}_2|},$$

where the last factor is from dividing by the flux, using that the particle density is $1/V$ (get V/V^2 for colliding two beams).

Note that this is relativistic. Write $dNdt = (d\sigma|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2)(Vdt)$, the LHS is the number of collisions, which should be the same in any frame, and the factor (Vdt) on the RHS is relativistically invariant. For simplicity, and with collider applications in mind, we're takeing $\vec{v}_1$ and $\vec{v}_2$ to be parallel, $\vec{v}_1 \times \vec{v}_2 = 0$ (otherwise replace $|\vec{v}_1 - \vec{v}_2|$ with $\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$). We want $d\sigma$ to be defined to be the cross section in the rest frame of one of the particles, so we want to define it to be boost invariant. So we need to show that $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is boost invariant; in the rest frame of particle 2 it reduces to $v_{rel}\rho_1\rho_2$, which is what we want. Let's just check it. Under a boost to a frame with relative velocity u (taken along the direction of $\vec{v}_1$ and $\vec{v}_2$, we have $v_i \to (v_i+u)/(1+v_i u)$ and $\rho_i \to \rho_i \gamma_u (1+v_i u)$, so $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is indeed invariant.

For our application, we define $\rho_i = 1/V$ in the lab frame.

Two body final states (in CM frame): $D = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^3(\vec{p}_1+\vec{p}_2)\delta(E_1 + E_2 - E_T)$ gives

$$D = \int \frac{1}{(2\pi)^3 4E_1E_2} p_1^2 dp_1 d\Omega_1 (2\pi)\delta(E_1+E_2-E_T).$$

Using $E_1 = \sqrt{p_1^2+m^2}$ and $E_2 = \sqrt{p_1^2+m^2}$ get $\partial(E_1+E_2)/\partial p_1 = p_1 E_T/E_1E_2$ and finally $D = p_1 d\Omega_1/16\pi^2 E_T$. This should be divided by 2! (more generally, n!) if the final states are identical.

- Example. For $\mu^2 > 4m^2$, consider $\phi \to \bar{N}N$ decay. $\mathcal{A} = -g$, and get

$$\Gamma = \frac{g^2}{2\mu}\frac{p_1}{16\pi^2\mu}\int d\Omega_1 = \frac{g^2}{8\pi\mu^2}\frac{\sqrt{\mu^2-4m^2}}{2},$$

where the last factor is p_1.

For $2 \to 2$ scattering in the CM frame,

$$d\sigma = \frac{|\mathcal{A}|^2}{4E_1E_2}\frac{p_f d\Omega_1}{16\pi^2 E_T}\frac{1}{|\vec{v}_1-\vec{v}_2|} = \frac{|\mathcal{A}|^2 p_f d\Omega_1}{64\pi^2 p_i E_T^2}$$

where we used $|\vec{v}_1 - \vec{v}_2| = p_1(E_1^{-1} + E_2^{-2}) = p_i E_T/E_1E_2$ in the CM frame, and p_i is the magnitude of the initial momentum, and p_f is that of the final momentum.

- Green functions $\widetilde{G}^{(n)}(p_1, \ldots, p_n)$, computed with external leg propagators, allowed to be off-shell. Can then compute e.g.

$$\langle k_3, k_4|S-1|k_1k_2\rangle = \prod_{n=1}^{4}\frac{k_n^2-m_n^2}{i}\widetilde{G}(-k_3,-k_4,k_1,k_2),$$

where the factors are to amputate the external legs. Consider for example $\widetilde{G}^{(4)}(k_1,k_2,k_3,k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon} + 2 \text{ permutations.}$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2 \to 2$ scattering using the above formula, because it is set to zero by $\prod n = 1^4(k_n^2 - m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop.