10/21 Lecture outline

⋆ **Reading: Luke, chapter 5-7**

• Last times, simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

For $NN \to NN$,

$$i\mathcal{A} = (-ig)^2\left(\frac{i}{(p_1 - p_1') - \mu^2} + \frac{i}{(p_1 - p_2') - \mu^2}\right).$$

$N(p_1) + \bar{N}(p_2) \to N(p_1') + \bar{N}(p_2')$ has

$$i\mathcal{A} = (-ig)^2\left(\frac{i}{(p_1 - p_1') - \mu^2} + \frac{i}{(p_1 + p_2) - \mu^2}\right).$$

$N(p_1) + \bar{N}(p_2) \to \phi(p_1')\phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2\left(\frac{i}{(p_1 - p_1') - m^2} + \frac{i}{(p_1 - p_2') - m^2}\right).$$

$N(p_1) + \phi(p_2) \to N(p_1') + \phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2\left(\frac{i}{(p_1 - p_2') - m^2} + \frac{i}{(p_1 + p_2) - m^2}\right).$$

• Mandelstam variables. $s = (p_1+p_2)^2$, $t = (p_1-p_1')^2$, $u = (p_1-p_2')^2$, with $s+t+u = 4m^2$ (more generally, $s+t+u = \sum_{i=1}^4 m_i^2$). In CM, $s = 4E^2$, $t = -2p^2(1-\cos\theta)$, and $u = -2p^2(1+\cos\theta)$.

• We saw that the t channel term above is associated with the Yukawa potential. The u channel term is similar. Now consider the s channel, in e.g. the $N + \bar{N}$ scattering amplitude. Using the CM relations $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $E_1 = E_2 = \sqrt{p^2 + m^2}$ gives

$$\mathcal{A} \sim \frac{1}{4m^2 + 4p^2 - \mu^2 + i\epsilon},$$

so for $\mu < 2m$ the denominator is always positive, and the amplitude decreases with increasing p^2. For $\mu > 2m$ there is a pole at $(p_1 + p_2)^2 = \mu^2$, where the intermediate meson goes on shell. This leads to a peak (not a pole, of course; because the intermediate particle is unstable anyway, the denominator gets an imaginary contribution), a *resonance*, in the cross section. E.g. Z_0 pole in $e^+e^- \to \mu^+\mu^-$, but not in $e^+e^- \to \gamma\gamma$.

• Crossing symmetry, CPT.

• Compute probabilities by squaring the S-maxtrix amplitudes, but have to be careful with the delta functions, since squaring the delta functions would give nonsense.

Consider quantum mechanics, with $U(t) = Te^{-i\int^t H(t)dt}$,

$$\langle f|U(t)|i\rangle \approx -i\langle f|H_{int}|i\rangle \int_0^t dte^{i\omega t},$$

where $\omega = E_f - E_i$. Squaring gives $P(t) = 2|\langle f|H_{int}|i\rangle|^2(1-\cos\omega t)/\omega^2$. For $t\to\infty$, multiply by $dE_f\rho(E_f)$ and replace $(1-\cos\omega t)/\omega^2 \to \pi t\delta(\omega)$ to get

$$\dot{P}_{i\to f} = 2\pi|\langle f|H_{int}|i\rangle|^2\rho(E).$$

Fermi's Golden Rule. But naively taking $t\to\infty$ initially would have given amplitude $\sim \delta(\omega)$, and squaring that would give $\delta(\omega)^2$, which needs to be replaced with $\delta(\omega)2\pi T$, and then divide by T to get the rate. Similarly in field theory, $\delta(p)^2$ should be replaced with probability $\sim \delta(p)$ times phase space volume factors.

• Phase space factors. Put the system in a box of volume V. The momenta are quantized and, as usual, there are $Vd^3\vec{k}/(2\pi)^3$ states with $\vec{k}$ in the range $d^3\vec{k}$. Interested in computing probabilities, $P = |\langle f|i\rangle|^2/\langle f|f\rangle\langle i|i\rangle$. Use e.g. $\langle k|k\rangle = (2\pi)^3 2\omega\delta^3(0)$ and replace $\delta^3(0)\to V$. Put these normalization factors into correct normalization of initial and final states:

$$\langle f|(S-1)|i\rangle_{VT} = i\mathcal{A}_{fi}^{VT}\delta^4(p_F - p_I)\prod_f \frac{1}{\sqrt{2\omega_k V}}\prod_i \frac{1}{\sqrt{2\omega_k V}},$$

where the factors account for the relativistic normalization of the states. Squaring, with the replacement $(2\pi^4\delta^4(p))^2 \to VT(2\pi)^4\delta^4(p)$ (since $\int d^4xe^{i0\cdot x} = VT$) get that the probability per unit time is

$$|\mathcal{A}_{fi}|^2 VD\prod_i \frac{1}{2E_i V},$$

where

$$D = (2\pi)^4\delta^4(p_F - p_I)\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}.$$

Decays: differential decay probability per unit time: $d\Gamma = \frac{1}{2M}|\mathcal{A}_{fi}|^2 D$. Integrate over all possible final states to get $\Gamma = 1/\tau$ where τ is the lifetime.

Cross sections: the number of scatterings per unit time is $dN = F d\sigma$, where F is the flux. So

$$d\sigma = \frac{\mathcal{A}_{fi}^2}{4E_1E_2V}\ D\ \frac{V}{|\vec{v}_1 - \vec{v}_2|},$$

where the last factor is from dividing by the flux, using that the particle density is $1/V$ (get V/V^2 for colliding two beams).

Note that this is relativistic. Write $dNdt = (d\sigma|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2)(Vdt)$, the LHS is the number of collisions, which should be the same in any frame, and the factor (Vdt) on the RHS is relativistically invariant. For simplicity we take $\vec{v}_1$ and $\vec{v}_2$ to be parallel, $\vec{v}_1 \times \vec{v}_2 = 0$. We want $d\sigma$ to be defined to be the cross section in the rest frame of one of the particles, so we want to define it to be boost invariant. So we need to show that $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is boost invariant; in the rest frame of particle 2 it reduces to $v_{rel}\rho_1\rho_2$, which is what we want. Let's just check it. Under a boost to a frame with relative velocity u (taken along the direction of $\vec{v}_1$ and $\vec{v}_2$, we have $v_i \to (v_i + u)/(1 + v_i u)$ and $\rho_i \to \rho_i\gamma_u(1 + v_i u)$, so $|\vec{v}_1 - \vec{v}_2|\rho_1\rho_2$ is indeed invariant.

For our application, we define $\rho_i = 1/V$ in the lab frame.