10/19 Lecture outline

⋆ **Reading: Luke, chapter 5**

• Last time, simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation. $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

$$= \left(e^{i(p_1'x_1+p_2'x_2)} + e^{i(p_1'x_2+p_2'x_1)}\right)\left(e^{-i(p_1x_1+p_2x_2)} + e^{-i(p_1x_2+p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1-p_1')^2-\mu^2} + \frac{1}{(p_1-p_2')^2-\mu^2}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2+m^2}, e\widehat{e})$ and $p_2 = (\sqrt{p^2+m^2}, -p\widehat{e})$, $p_1' = (\sqrt{p^2+m^2}, p\widehat{e}')$, $p_2' = (\sqrt{p^2+m^2}, -p\widehat{e}')$, where $\widehat{e}\cdot\widehat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1-\cos\theta)+\mu^2} + \frac{1}{2p^2(1+\cos\theta)+\mu^2}\right).$$

As we'll discuss, scattering by ϕ exchange leads to an attractive Yukawa potential.

• Feynman diagrams. Each vertex gets $(-ig)(2\pi)^4\delta^4(p_{total\ in})$, each internal line gets $\int\frac{d^4k}{(2\pi)^4}D_F(k^2)$, where D_F is the propagator, e.g. $D_F(k^2) = \frac{i}{k^2-m^2+i\epsilon}$. Result is $\langle f|(S-1)|i\rangle$, so divide by $(2\pi)^4\delta^4(p_F - p_I)$ to get $i\mathcal{A}_{fi}$.

If the diagram has no loops, the momentum conserving delta functions fix all internal momenta in terms of the external ones. When the diagram has $L \neq 0$ loops, the procedure above yields integrals over the internal momenta of the loops. (Note that if a diagram has I internal lines and V vertices, then there are I momentum integrals, and V momentum conserving delta functions; one of these becomes overall momentum conservation, so there are $L = I - V - 1$ momentum integrals left to do, and L is the number of loops in the diagram.) Any loop momentum integrals require renormalization, which we'll discuss later (next quarter), so for now we'll just consider "tree-level" contributions, associated with diagrams without loops.

• More examples:

(1) $N(p_1) + \bar{N}(p_2) \to N(p_1') + \bar{N}(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - \mu^2} + \frac{i}{(p_1 + p_2) - \mu^2} \right).$$

(2) $N(p_1) + \bar{N}(p_2) \to \phi(p_1')\phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - m^2} + \frac{i}{(p_1 - p_2') - m^2} \right).$$

(3) $N(p_1) + \phi(p_2) \to N(p_1') + \phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_2') - m^2} + \frac{i}{(p_1 + p_2) - m^2} \right).$$

• Mandelstam variables. $s = (p_1 + p_2)^2$, $t = (p_1 - p_1')^2$, $u = (p_1 - p_2')^2$, with $s + t + u = 4m^2$ (more generally, $s + t + u = \sum_{i=1}^4 m_i^2$). In CM, $s = 4E^2$, $t = -2p^2(1 - \cos\theta)$, and $u = -2p^2(1 + \cos\theta)$.

• Yukawa potential. Indeed, the first term in e.g. the above $N + N$ scattering amplitude gives, upon using $(p_1 - p_1')^2 - \mu^2 = -(|\vec{p}_1 - \vec{p}_1'|^2 + \mu^2)$, and the Born approximation in NRQM, $\mathcal{A}_{NR} = -i \int d^3\vec{r} e^{-i(\vec{p}' - \vec{p})} U(\vec{r})$, the attractive Yukawa potential

$$U(r) = \int \frac{d^3p}{(2\pi)^3} \frac{-(g/2m)^2 e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \mu^2} = -\frac{(g/2m)^2}{4\pi r} e^{-\mu r}.$$

(The $1/(2m)^2$ is because we normalized the relativistic states with the extra factor of $2\omega_k \approx 2m$ as compared with standard nonrelativistic normalization.) This gives Yukawa's explanation of the attraction between nucleons, from meson exchange.