10/12 Lecture outline

⋆ **Reading: Luke, chapter 5**

• Last time: left off discussing the interaction picture: writing $H = H_0 + H_{int}$, in this picture we use H_0 to time evolve the operators, and H_{int} to time evolve the states:

$$i\frac{d}{dt}\mathcal{O}(t) = [\mathcal{O}, H_0], \qquad i\frac{d}{dt}|\psi(t)\rangle = H_{int}|\psi(t)\rangle.$$

For example, we'll take H_0 to be the free Hamilton of KG fields, with only the mass terms included in the potential. Again, this is free because the EOM are linear, and we can solve for $\phi(x)$ by superposition. As before, upon quantization, the fields become superpositions of creation and annihilation operators. The states are all the various multiparticle states, coming from acting with the creation operators on the vacuum.

• Simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Derive it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does.

Now use Wick's theorem:

$$T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + \sum_{contractions} : \phi_1 \ldots \phi_n :$$

to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3x g\phi\psi^\dagger\psi$. Use $\phi \sim a + a^\dagger$ for "mesons," $\psi \sim b + c^\dagger$, and $\psi^\dagger \sim b^\dagger + c$. We'll say that b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. Conservation law, conserved charge $Q = N_b - N_c$.

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$.

Now consider $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

$$= \left(e^{i(p_1'x_1 + p_2'x_2)} + e^{i(p_1'x_2 + p_2'x_1)}\right)\left(e^{-i(p_1x_1 + p_2x_2)} + e^{-i(p_1x_2 + p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1 - p_1')^2 - \mu^2} + \frac{1}{(p_1 - p_2')^2 - \mu^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2 + m^2}, e\widehat{e})$ and $p_2 = (\sqrt{p^2 + m^2}, -p\widehat{e})$, $p_1' = (\sqrt{p^2 + m^2}, p\widehat{e}')$, $p_2' = (\sqrt{p^2 + m^2}, -p\widehat{e}')$, where $\widehat{e} \cdot \widehat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1 - \cos\theta) + \mu^2} + \frac{1}{2p^2(1 + \cos\theta) + \mu^2}\right).$$

The scattering by ϕ exchange leads to an attractive Yukawa potential. Indeed, the first term in the above amplitude gives, upon using $(p_1 - p_1')^2 - \mu^2 = |\vec{p}_1 + \vec{p}_1'|^2 + \mu^2$, and the Born approximation in NRQM, $\mathcal{A}_{NR} = -i\int d^3\vec{r} e^{-i(\vec{p}\,'-\vec{p})} U(\vec{r})$, the attractive Yukawa potential

$$U(r) = \int \frac{d^3p}{(2\pi)^3} \frac{-g^2 e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2} = -\frac{g^2}{4\pi r} e^{-\mu r}.$$

This gives Yukawa's explanation of the attraction between nucleons, from meson exchange.

• Feynman diagrams. Each vertex gets $(-ig)(2\pi)^4\delta^4(p_{total\ in})$, each internal line gets $\int \frac{d^4k}{(2\pi)^4} D_F(k^2)$, where D_F is the propagator, e.g. $D_F(k^2) = \frac{i}{k^2-m^2+i\epsilon}$. Result is $\langle f|(S-1)|i\rangle$, so divide by $(2\pi)^4\delta^4(p_F - p_I)$ to get $i\mathcal{A}_{fi}$.

If the diagram has no loops, the momentum conserving delta functions fix all internal momenta in terms of the external ones. When the diagram has $L \neq 0$ loops, the procedure above yields integrals over the internal momenta of the loops. These integrals require renormalization, which we'll discuss later (next quarter), so for now we'll just consider "tree-level" contributions, associated with diagrams without loops.

• More examples:

(1) $N(p_1) + \bar{N}(p_2) \to N(p_1') + \bar{N}(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - \mu^2} + \frac{i}{(p_1+p_2) - \mu^2} \right).$$

(2) $N(p_1) + \bar{N}(p_2) \to \phi(p_1')\phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1') - m^2} + \frac{i}{(p_1+p_2) - m^2} \right).$$

(3) $N(p_1) + \phi(p_2) \to N(p_1') + \phi(p_2')$ has

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_2') - m^2} + \frac{i}{(p_1+p_2) - m^2} \right).$$

• Phase space factors. Put the system in a box of volume V. The momenta are quantized and, as usual, there are $Vd^3\vec{k}/(2\pi)^3$ states with $\vec{k}$ in the range $d^3\vec{k}$. Also, normalizing ϕ and $a(k)$ and $a^\dagger(k)$ yields

$$\langle f|(S-1)|i\rangle_{VT} = i\mathcal{A}_{fi}^{VT}\delta^4(p_F - p_I) \prod_f \frac{1}{\sqrt{2\omega_k V}} \prod_i \frac{1}{\sqrt{2\omega_k V}}.$$

Squaring, get that the probability per unit time is

$$|\mathcal{A}_{fi}|^2 V D \prod_i \frac{1}{2E_i V},$$

where

$$D = (2\pi)^4 \delta^4(p_F - p_I) \prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}.$$

Differential decay probability per unit time: $d\Gamma = \frac{1}{2M}|\mathcal{A}_{fi}|^2 D$.