10/5 Lecture outline

$\star$ **Reading: Luke, chapters 3 and 4. Maybe a bit of chapter 5, if time.**

- Aside on symmetries of $\mathcal{L}$ and Noether's theorem. If a variation $\delta\phi_a$ changes $\delta L = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu \partial_\nu \phi_a$, $\delta\mathcal{L} = \epsilon^\nu \partial_\nu \mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\partial_\nu \phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x} T_{\mu 0}$.

Another example: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.

Another example: $\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - \mu^2 \psi^\dagger \psi$, has symmetry under $\psi \to e^{i\alpha}\psi$. $Q =$ (HW).

- Continue with quantization of the KG field theory example.

$$[\phi_a(\vec{x}, t), \Pi_b(\vec{y}, t)] = i\delta_{ab}\delta^3(\vec{x} - \vec{y}) \quad \textit{(Equal time commutators).}$$

Write

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}].$$

Then canonical quantization implies that

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3(2\omega)\delta^3(\vec{k} - \vec{k}'),$$

creation and annihilation operators. The quantum field is a superposition of creation and annihilation operators. Note also that

$$H = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})).$$

Need to normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum.

- Recall $[\phi(x), \phi(y)] = D(x - y) - D(y - x)$, where

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x - y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

For spacelike separation, $(x-y)^2 = -r^2$, get $D(x-y) = \frac{m}{2\pi^2 r} K_1(mr)$. Although $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone, the above difference is not. Good.

- Get more interesting theories by adding interactions, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$, treat 2nd term as a perturbation.
- Consider green's functions for the KG equation,

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y),$$

which we can use to solve $(\partial_x^2 + m^2)\phi(x) = \rho(x)$, via $\phi(x) = \phi_0(x) + i\int d^4y D(x-y)\rho(y)$, where ϕ_0 is a solution of the homogeneous KG equation. Get

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2+m^2}$. There are choices about whether the contour goes above or below the poles. Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. For $x_0 > y_0$, note that

$$D_R = \int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)}) \equiv D(x-y) - D(y-x) = \langle[\phi(x), \phi(y)]\rangle.$$

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

- Feynman propagator. Define

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle\phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle\phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\ldots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f|t_i\rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. This ensures causality.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y))- : A(x)B(y) :$. This is a number, not an operator, e.g. for $x^0 > y^0$ the contraction is $[A^+, B^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.

• Simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x\mathcal{H}_I}|i\rangle.$$

Now use Wick's theorem:

$$T(\phi_1\ldots\phi_n) =: \phi_1\ldots\phi_n : + \sum_{contractions} : \phi_1\ldots\phi_n :$$

to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the above theory, with $H_{int} = \int d^3xg\phi\psi^\dagger\psi$. Use $\phi \sim a+a^\dagger$ for "mesons," $\psi \sim b+c^\dagger$ for "nucleons," and $\psi^\dagger \sim b^\dagger+c$ for the anti-nucleons. Conservation law, conserved charge $Q = N_c - N_b$.

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$.

Now consider $\psi\psi \to \psi\psi$, to $\mathcal{O}(g^2)$. Get

$$T\frac{-ig)^2}{2!}\int d^4x_1d^4x_2\phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$
$$= \left(e^{i(p_1'x_1+p_2'x_2)} + e^{i(p_1'x_2+p_2'x_1)}\right)\left(e^{-i(p_1x_1+p_2x_2)} + e^{-i(p_1x_2+p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1-p_1')^2 - m^2} + \frac{1}{(p_1-p_2')^2 - m^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1'p_2').$$

(Associated with $U_{Yuk} \sim e^{-mr}/r$.)

We'll connect with Feynman diagrams.