9/30 Lecture outline

⋆ **Reading: Luke, chapters 3 and 4. Maybe a bit of chapter 5, if time.**

• Last time: classical field theory. E.g. scalars $\phi_a(t,\vec{x})$, with $S=\int d^4x\mathcal{L}(\phi_a,\partial_\mu\phi_a)$. Then $\Pi^\mu_a=\partial\mathcal{L}/\partial(\partial_\mu\phi_a)$, and E.L. eqns $\partial\mathcal{L}/\partial\phi_a=\partial_\mu\Pi^\mu_a$. Define $\Pi_a\equiv\Pi^0_a$. $H=\int d^3x(\Pi\dot{\phi}_a-\mathcal{L})=\int d^3x\mathcal{H}$.

Example: $\mathcal{L}=\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2)$. Klein-Gordon equation, $(\partial^2+m^2)\phi=0$. Can't interpret ϕ as a probability wavefunction because of solutions $E=\pm\sqrt{\vec{p}^2+m^2}$. But we'll see that the KG equation is fine as a field theory. The field has both creation and annihilation operators, corresponding to the $E=\pm\sqrt{\vec{k}^2+m^2}$ solutions. Write general classical solution

$$\phi_{cl}(x)=\int\frac{d^3k}{(2\pi)^3(2\omega(k))}[a_{cl}(k)e^{-ikx}+a^*_{cl}(k)e^{ikx}],$$

where $a_{cl}(k)$ are classical constants of integration, determined by the initial conditions. We'll quantize soon.

Another example: $\mathcal{L}=\frac{i}{2}(\psi^*\dot{\psi}-\dot{\psi}^*\psi)-\nabla\psi^*\cdot\nabla\psi-m\psi^*\psi$. Get EOM: $i\partial_t\psi=-\nabla^2\phi+m\psi$. Looks like S.E., but again don't want to interpret ψ as a probability amplitude – here it's a field, that we can consider quantizing. This example won't work for ψ a scalar field, but we'll later consider an analogous theory where ψ is a fermion field, and the equation is the Dirac equation.

• Canonical quantization: generalize QM by replacing $q_a(t)\to\phi_a(t,\vec{x})$. QM is like QFT in zero spatial dimensions, with the field playing role of position before:

$$[\phi_a(\vec{x},t),\Pi_b(\vec{y},t)]=i\delta_{ab}\delta^3(\vec{x}-\vec{y})\quad\textit{(Equal time commutators).}$$

• Consider the KG equation in $0+1$ dimensions, i.e. the SHO: $L=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}\omega^2x^2$, $p=\partial L/\partial\dot{\phi}=\dot{\phi}$. Classical EOM solved by $x_{cl}=ae^{-i\omega t}+a^*e^{i\omega t}$. Now quantize: $[x,p]=i\hbar$, $[a,a^\dagger]=1$, $H=\omega(a^\dagger a+\frac{1}{2})$. In the Heisenberg picture, $\widehat{x}=\sqrt{\frac{1}{2\omega}}(ae^{-i\omega t}+a^\dagger e^{i\omega t})$; $p=\dot{x}=i\sqrt{\frac{\omega}{2}}(ae^{i\omega t}-a^\dagger e^{-i\omega t})$.

• Now quantize the KG field theory in $3+1$ dimensions. Write

$$\phi(x)=\int\frac{d^3k}{(2\pi)^3(2\omega(k))}[a(k)e^{-ikx}+a^\dagger(k)e^{ikx}].$$

Then canonical quantization implies that

$$[a(\vec{k}),a^\dagger(\vec{k}')]=(2\pi)^3(2\omega)\delta^3(\vec{k}-\vec{k}'),$$

creation and annihilation operators. The quantum field is a superposition of creation and annihilation operators. Note also that

$$H = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})).$$

Need to normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum.

• Causality? Compute $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$, where

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note that the commutator is a c-number, not an operator. For spacelike separation, $(x-y)^2 = -r^2$, get $D(x-y) = \frac{m}{2\pi^2 r}K_1(mr)$. Although $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone, the above difference is not. Good.

• Get more interesting theories by adding interactions, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4$, treat 2nd term as a perturbation.

• Symmetries of $\mathcal{L}$ and Noether's theorem. If a variation $\delta\phi_a$ changes $\delta L = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu\partial_\nu\phi_a$, $\delta\mathcal{L} = \epsilon^\nu\partial_\nu\mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x}T_{\mu 0}$.

Another example: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.

• Consider green's functions for the KG equation,

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y),$$

which we can use to solve $(\partial_x^2 + m^2)\phi(x) = \rho(x)$, via $\phi(x) = \phi_0(x) + i\int d^4y D(x-y)\rho(y)$, where ϕ_0 is a solution of the homogeneous KG equation. Get

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2 + m^2}$. There are choices about whether the contour goes above or below the poles. Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. For $x_0 > y_0$, note that

$$D_R = \int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)}) \equiv D(x-y) - D(y-x) = \langle [\phi(x), \phi(y)] \rangle.$$

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator. Define

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle \phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle \phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\ldots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f | t_i \rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. This ensures causality.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator, e.g. for $x^0 > y^0$ the contraction is $[A^+, B^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$.

• Simple example of interacting theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi^2 - \mu^2\phi^2) + (\partial\psi^\dagger\partial\psi - m^2\psi^\dagger\psi) - g\phi\psi\psi^\dagger.$$

Toy model for interacting nucleons and mesons. Treat last term as a perturbation.

• Dyson's formula. Compute scattering S-matrices. Consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle.

$$|\psi(t)\rangle = Te^{-i\int d^4x \mathcal{H}_I}|i\rangle.$$

Now use Wick's theorem:

$$T(\phi_1 \ldots \phi_n) =: \phi_1 \ldots \phi_n : + \sum_{contractions} : \phi_1 \ldots \phi_n :$$

to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

Next time, look at some examples, and connect with Feynman diagrams.