12/2 Lecture outline

⋆ **Reading: Luke chapter 11. Tong chapter 6**

• Last time: massive spin 1 field, with

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mu^2 A_\mu A^\mu,$$

which gives the EOM $\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0$, and this implies $\partial_\mu A^\mu = 0$; the vector is transverse, by construction. We quantized the spatial components

$$[A_i(t,\vec{x}), F^{j0}(t,\vec{y})] = i\delta_i^j \delta^{(3)}(\vec{x}-\vec{y})$$

and obtained the Feynman rules that massive vectors have the momentum space propagator

$$\left[\frac{-i(g_{\mu\nu} - k_\mu k_\nu/\mu^2)}{k^2 - \mu^2 + i\epsilon}\right].$$

And $\langle 0|A_\mu(x)|V(k,r)\rangle = \epsilon_\mu(k)^r e^{-ikx}$, so incoming vector mesons have $\epsilon^r_\mu(k)$ and outgoing have $\epsilon^{*r}(k)$.

We can couple the massive vector to other fields, e.g. to a fermion via $\mathcal{L}_{int} = -g\bar{\psi}\not{A}\Gamma\psi$, with $\Gamma = 1$ (vector) or $\Gamma = \gamma_5$ (axial vector). Gives Feynman rule that a vertex has a factor of $-ig\gamma^\mu\Gamma$.

• Now consider the massless theory. If we add $\mathcal{L} \supset -A_\mu j^\mu$ to the massive theory, get $\partial_\mu A^\mu = \mu^{-2}\partial_\mu j^\mu$, so there is only a sensible limit if $\partial_\mu j^\mu = 0$, must couple to a conserved current. Associate with symmetry, $\psi \to e^{-i\lambda q}\psi$, where q is the charge. The massless theory must be associated with gauge invariance: can make above symmetry transformations where $\lambda = \lambda(x)$ is a local function, and this is a redundancy, rather than a symmetry, when combined with $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\lambda(x)$, where e is a coupling constant. Consider minimal coupling: replace $\partial^\mu \to D^\mu = \partial^\mu + ieA^\mu q$ for a charge q field to ensure that the theory respects gauged version of the symmetry.

Another way to say it: the only way to have a sensible $\mu \to 0$ limit is if A_μ is a gauge field, associated with a local gauge symmetry. The reason is that the operator in brackets in

$$[\eta_{\mu\nu}(\partial^\rho\partial_\rho) - \partial_\mu\partial_\nu]A^\nu = 0$$

is not invertable: it annihilates any function of form $\partial_\mu\lambda$. Solution: require that $A_\mu \sim A_\mu + \partial_\mu\lambda$, i.e. gauge invariance. The space of gauge fields has equivalent gauge orbits.

Minimal coupling examples:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - eq\not{A} - m)\psi.$$

$$\mathcal{L} = D_\mu\phi^* D^\mu\phi - m^2|\phi|^2.$$

The first gives a $\bar{\psi}A_\mu\psi$ Feynman vertex weighted by $-ieq\gamma^\mu$, and the second gives a $\phi^*(p')A_\mu\phi(p)$ vertex weighted by $ieq(p+p')^\mu$, along with a $A_\mu A_\nu\phi^*\phi$ seagull graph weighted by $2ie^2q^2g^{\mu\nu}$ (factor of 2 because of the two identical A_μ fields).

As in the massive vector case, A_0 has no kinetic term, can solve its EOM ($\nabla\cdot\vec{E} = 0 \to \nabla^2 A_0 + \nabla\cdot\dot{\vec{A}} = 0$):

$$A_0(\vec{x}) = \int d^3\vec{x}' \frac{\nabla\cdot\dot{\vec{A}}(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}.$$

Gauge fixing: can always choose e.g. $\partial_\mu A^\mu = 0$. Doesn't entirely fix the gauge. Can still pick $\nabla\cdot\vec{A} = 0$ – Coulomb gauge – then $A_0 = 0$. See two polarizations. So take $\vec{\epsilon}^r$ with $\vec{\epsilon}_r\cdot\vec{p} = 0$, orthonormal. The completeness relation is similar to that above, except that we replace $\mu^2 \to |\vec{p}|^2$. The propagator is then

$$\langle TA_i(x)A_j(y)\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{i(\delta_{ij} - k_ik_j/|\vec{k}|^2)}{k^2 + i\epsilon}\right].$$

This gauge can be a pain in the interacting theory (need to write instantaneous $\delta(x^0 - y^0)/|\vec{x}-\vec{y}|$ Coulomb interaction). It's nicer to write something more manifestly Lorentz invariant.

In the massive vector case, we had the propagator $-i(g_{\mu\nu} - k_\mu k_\nu/|\mu|^2)/(k^2 - \mu^2 + i\epsilon)$. In the $\mu \to 0$ massless gauge theory, gauge invariance ensures that the $k_\mu k_\nu$ term has no effect in physical, on-shell amplitudes. For example, $e^+e^- \to \mu^+\mu^-$ tree-level amplitude, show that the $k_\mu k_\nu$ term in the propagator doesn't contribute for on-shell external states. Another example: Compton scattering of vector off an electron: $i\mathcal{A} = \mathcal{M}^{\mu\nu}\epsilon_\mu^{(r')*}(k')\epsilon_\nu^{(r)}(k)$. Observe that $k^\mu\mathcal{M}_{\mu\nu} = 0$, decouples the helicity 0 mode. Also, square amplitude and average over initial polarizations and sum over the final ones, and note that $k^\mu\mathcal{M}_{\mu\nu}$, and likewise for k', ensures that the $1/\mu^2$ terms in the polarization completeness relation go away.

• Gauge fixing. Try to preserve Lorentz invariance by imposing $\partial_\mu A^\mu = 0$, and not $A_0 = 0$. Can modify $\mathcal{L}$ to get Lorentz gauge EOM. More generally, can consider

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial\cdot A)^2,$$

and quantize for any parameter α. Popular choices are $\alpha = 1$ (Feynman) and $\alpha = 0$ (Landau). Now get $\pi^0 = \partial\mathcal{L}/\partial(\dot{A}_0) = -\partial_\mu A^\mu/\alpha$. Do canonical quantization for all components, $[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = i\eta_{\mu\nu}\delta(\vec{x} - \vec{y})$. Write plane wave expansion with 4 polarizations, normalized to $\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}$. Get that timelike polarizations create negative norm states. Can fix this by imposing $\partial^\mu A_\mu^+|\Psi\rangle = 0$ on the physical states, along with gauge invariance relation, to get a physical Hilbert space with positive norms.

Propagator for gauge field is

$$\langle TA_\mu(x)A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{-i(g_{\mu\nu} + (\alpha - 1)k_\mu k_\nu/k^2)}{k^2 + i\epsilon}\right].$$

Again, the $k_\mu k_\nu$ piece will drop out in the end in physical amplitudes. Just need to make a choice and stick with it consistently. Or keep α as a parameter, and then it's a good check on the calculation that the α indeed drops out in the end.

- QED examples:

$e^+e \to \gamma\gamma$:

$e^+e^- \to e^+e^-$:

$e^-\gamma \to e^-\gamma$:

$e^-e^\mp \to e^-e^\mp$ and the Coulomb potential. Contrast with scalar Yukawa case, where the potential is always attractive, whereas here opposites attract while like charges repel. Because here $\bar{v}\gamma^0 v \to +2m$, whereas in the scalar case got $\bar{v}v \to= -2m$.