11/16 Lecture outline

⋆ **Reading: Luke chapter 9. Tong chapter 4**

• Recall, the Dirac action:

$$S = \int d^4x \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)$$
$$= \int d^4x(u_+^\dagger i\sigma^\mu \partial_\mu u_+ + u_-^\dagger i\bar{\sigma}^\mu \partial_\mu u_- - m(u_+^\dagger u_- + u_-^\dagger u^+)).$$

Let's first consider the plane wave solutions for a single Wey spinor $u_+$, in the $m = 0$ case, so the EOM is $\partial_\mu \sigma^\mu u_+(x) = 0$. Take positive energy, $k_0 = +\sqrt{\vec{k}^2}$, and then the plane wave solutions are

$$u_+(x) = u_+ e^{-ikx}, \qquad \text{or} \qquad u_+(x) = v_+ e^{ikx}.$$

When we quantize, $u_+$ will go with a particle annihilation operator, and $v_+$ will go with an antiparticle creation operator. Plugging into the EOM, $(k_0 - \vec{\sigma}\cdot\vec{k})u_+ = 0$. Taking $\vec{k} = k_0\hat{z}$, get

$$\langle 0|u_+(x)|k\rangle \propto e^{-ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note also that the state $|k\rangle$ has spin $J_z = 1/2$, under a rotation by $\theta$ around the $\widehat{z}$ axis, it picks up a phase $e^{i\theta/2}$. The state $|k\rangle$ thus carries helicity $+1/2$, and the annihilation operator that goes with $u_+$ annihilates that state. Likewise

$$\langle k|v_+(x)|0\rangle \propto e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so $v_+$ goes with a creation operator creating states of angular momentum $-1/2$ along the direction of motion. The theory has particles of helicity $1/2$ and antiparticles of helicity $-1/2$. This can only happen for massless fermions, since otherwise could get the opposite helicity in a boosted frame.

The plane wave solutions of the Dirac equation are

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

where

$$(\gamma^\mu p_\mu - m)u^s(p) = 0, \qquad (\gamma_\mu p^\mu + m)v^r(p) = 0.$$

The important properties are that these form a complete, orthogonal basis, with

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

For example, we can take in the rest frame of a massive fermion,

$$u^{(1)} = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

which can be boosted to get the solution for general $p^\mu$. This solution can be written e.g. as

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and $r, s$ label two independent basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The general solution of the classical EOM is a superposition of these plane waves:

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \left(b^r(p)u^r(p)e^{-ipx} + c^{r\dagger}(p)v^r(p)e^{ipx}\right)$$

The theory is quantized by using $\Pi^0_\psi = \partial\mathcal{L}/\partial(\partial_0\psi) = i\psi^\dagger$ and imposing

$$\{\psi(t,\vec{x}), \psi^\dagger(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}).$$

If we quantize with a commutator rather than anticommutator, get a Hamiltonian that is unbounded below, with $c$ creating antiparticles with negative energy. Shows that spin $\frac{1}{2}$ must have fermionic statistics, to avoid unitarity problems.

So the coefficients in the plane wave expansion get quantized to be annihilation and creation operators as

$$\{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}\delta^3(\vec{p}-\vec{p}'), ..$$

etc.

- Fermi statistics, Pauli exclusion.