11/11 Lecture outline

⋆ **Reading: Luke chapter 9. Tong chapter 4**

• Aside: consider $\sigma^\mu = (1, \sigma^i)$, where each entry is a 2×2 matrix. Now form $X = x^\mu \sigma^m u$. Lorentz transformations act as $X \to X' = DXD^\dagger$, where $D \in SL(2,C)$. Here $D = e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}$ for a rotation by θ around the $\widehat{e}$ axis, and $D_\pm = e^{\pm\vec{\sigma}\cdot\hat{e}\phi/2}$ for a boost along the $\widehat{e}$ axis, where $v = \tanh\phi$. This illustrates the statement of last lecture, that the vector representation of the Lorentz group is $D^(1/2, 1/2)$.

• Last time: $u_\pm$, with $D = e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}$ for a rotation by θ around the $\widehat{e}$ axis, and $D_\pm = e^{\pm\vec{\sigma}\cdot\hat{e}\phi/2}$ for a boost along the $\widehat{e}$ axis, where $v = \tanh\phi$. These 2-component Weyl spinor representations individually play an important role in non-parity invariant theories, like the weak interactions. Parity ($(t, \vec{x}) \to (t, -\vec{x})$) exchances them. So, in parity invariant theories, like QED, they are combined into a 4-component Dirac spinor, $(1/2, 0) + (0, 1/2)$:

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

The 4-component spinor rep starts with the clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, e.g.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

There are other choices of reps of the clifford algebra.

$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\eta^{\mu\nu}$, satisfies the Lorentz Lie algebra relation. Under a rotation, $S^{ij} = -\frac{i}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$, so taking $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ get under rotations

$$S[\vec{\varphi}] = \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

Under boosts, $\Omega_{i,0} = \phi_i$,

$$S[\Lambda] = \begin{pmatrix} e^{\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

This exhibits the 2-component reps that we described above.

Under Lorentz transformations, spinors transform as $\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$, and $\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger$. Note that $S[\Lambda]^\dagger S[\Lambda] \neq 1$, but $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma_0$. So define $\bar{\psi}(x) \equiv \psi^\dagger\gamma^0$ and note that $\bar{\psi}\psi$ transforms as a scalar, and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

For 2-component spinors, $u_-^\dagger \sigma^\mu u_-$ and $u_+^\dagger \bar{\sigma}^\mu u_+$ transform like vectors, where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Here are two Lorentz scalars (exchanged under parity): $u_\pm^\dagger u_\mp$.

$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$, anticommutes with all other γ^μ and $(\gamma^5)^2 = 1$. In our above representation of the gamma matrices, $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are projection operators, projecting on to $u_\pm$.

• The Dirac action:

$$S = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$
$$= \int d^4x (u_+^\dagger i\sigma^\mu \partial_\mu u_+ + u_-^\dagger i\bar{\sigma}^\mu \partial_\mu u_- - m(u_+^\dagger u_- + u_-^\dagger u^+)).$$

The last line exhibits something interesting: if there is a mass term, it is necessary to have both u_+ and u_- (and then there's parity invariance). But if $m = 0$, we can consider P non-invariant theories with only u_+ or only u_-. More about this soon. Also, the action has a global $U(1)$ symmetry under $\psi \to e^{i\alpha}\psi$, whose Noether conserved charge is fermion number. If $m = 0$, this symmetry is enhanced to $U(1)_+ \times U(1)_-$, acting separately on u_+ and u_-. Neat point: this enhanced symmetry helps explains why the known fermion masses are small.

Vary w.r.t. $\bar{\psi}$ to get the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Dirac wrote this down by thinking about how to make sense of the square-root of the operator appearing in the KG equation, $\sqrt{\partial_\mu \partial^\mu + m^2}$; indeed, $-(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m) = \partial^2 + m^2$.

The conjugate momentum to ψ is

$$\pi_\psi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu.$$

So ψ has 4 (rather than 8) real d.o.f..