11/9 Lecture outline

⋆ **Reading: Luke chapter 9. Tong chapter 4**

• On to fermions! Consider more generally Lorentz transformations. Under lorentz transformations $x^\mu \to x^{\mu'} = \Lambda^\mu_\nu x^\nu$, scalar fields transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. Vector fields transform as $A^\mu \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. Generally, $\phi^a \to D[\Lambda]^a_b \phi^b(\Lambda^{-1}x)$, where $D[\Lambda]$ is a rep of the Lorentz group, $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$. Write $D[\Lambda] = \exp(\frac{1}{2}\Omega_{\mu\nu}\mathcal{M}^{\mu\nu})$, which is a rep if $\mathcal{M}^{\nu\nu}$ satisfies the Lie algebra commutation relation $[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\mu\nu}] = \eta^{\sigma\mu}\mathcal{M}^{\rho\nu} \pm 3perms$, where the perms account for $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$. E.g. the fundamental rep has $(\mathcal{M}^{\mu\nu})^{\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}$.

Write the Lorentz transformation generators in terms rotation, whose generators are the angular momentum $\vec{J}$, where $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$, and boosts, with $\vec{K}$ and $K_i = M^{i,0}$. They are similar, e.g. boosting along the x axis vs rotation around the x axis:

$$\Lambda_{boost} = \begin{pmatrix} \cosh\phi & \sinh\phi & & \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \Lambda_{rotate} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{pmatrix}.$$

So define $\vec{N}^\pm \equiv \frac{1}{2}(\vec{J} \pm i\vec{K})$. Then the Lorentz algebra becomes simply $[N_i^\pm, N_k^\pm] = i\epsilon_{ijk}N_k^\pm$, and $[N^\pm, N_j^\mp] = 0$, i.e. two copies of the familiar rotation commutation relations. The reps are then labeled by (n_L, n_R), where n_L and n_R are non-negative half-integers, like the angular momentum j. Note that parity exchanges $\vec{N} \leftrightarrow \vec{N}^\dagger$, so it exchanges the above left and right, hence their names. The angular momentum $\vec{J} = \vec{N} + \vec{N}^\dagger$, so j runs from $|n_L - n_R|$ to $n_L + n_R$ The scalar rep is $(0,0)$, the vector rep is $(1/2, 1/2)$. The basic spinor reps are $(1/2, 0)$ and $(0, 1/2)$, denoted $u_\pm$; these are called left and right handed Weyl spinors. They both have $D = e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}$ for a rotation by θ around the $\hat{e}$ axis, but they have $D_\pm = e^{\pm\vec{\sigma}\cdot\hat{e}\phi/2}$ for a boost along the $\hat{e}$ axis, where $v = \tanh\phi$. These 2-component Weyl spinor representations individually play an important role in non-parity invariant theories, like the weak interactions. Parity $((t, \vec{x}) \to (t, -\vec{x}))$ exchances them. So, in parity invariant theories, like QED, they are combined into a 4-component Dirac spinor, $(1/2, 0) + (0, 1/2)$:

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

The 4-component spinor rep starts with the clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, e.g.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

(There are other choices of reps of the clifford algebra.)

Then consider $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\eta^{\mu\nu}$, and note that $S^{\mu\nu}$ satisfies the Lorentz Lie algebra relation. It's a spinor rep since it is easily verified that $\psi^\alpha \to -\psi^\alpha$ under a 2π rotation. Under a rotation, $S^{ij} = -\frac{i}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$, so taking $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ get under rotations

$$S[\vec{\varphi}] = \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

Under boosts, $\Omega_{i,0} = \chi_i$,

$$S[\Lambda] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

This exhibits the 2-component reps that we described above.

Under Lorentz transformations, spinors transform as $\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$, and $\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger$. Note that $S[\Lambda]^\dagger S[\Lambda] \neq 1$, but $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma_0$. So define $\bar{\psi}(x) \equiv \psi^\dagger\gamma^0$ and note that $\bar{\psi}\psi$ transforms as a scalar, and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

For 2-component spinors, $u_-^\dagger\sigma^\mu$ and $u_+^\dagger\bar{\sigma}^\mu u_+$ transform like vectors, where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Here are two Lorentz scalars (exchanged under parity): $u_\pm^\dagger u_\mp$.

$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$, anticommutes with all other γ^μ and $(\gamma^5)^2 = 1$. In our above representation of the gamma matrices, $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are projection operators, projecting on to $u_\pm$.

- The Dirac action:

$$S = \int d^4x \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x).$$

Vary w.r.t. $\bar{\psi}$ to get the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Dirac wrote this down by thinking about how to make sense of the square-root of the operator appearing in the KG equation, $\sqrt{\partial_\mu\partial^\mu + m^2}$; indeed, $-(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m) = \partial^2 + m^2$.

The conjugate momentum to ψ is

$$\pi_\psi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu.$$

So ψ has 4 (rather than 8) real d.o.f..

The plane wave solutions of the Dirac equation are

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

where

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and r, s label two independent basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The general solution of the classical EOM is a superposition of these plane waves. We'll form these, and then quantize.