11/2 Lecture outline

⋆ **Reading: Tong 3.7, Luke, chapter 8**

• Last time: Greens functions.

$$G^{(n)}(x_1, \dots x_n) = \langle \Omega | T\phi_H(x_1) \dots \phi_H(x_n) | \Omega \rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian.

Showed that

$$G^{(n)}(x_1 \dots x_n) = \frac{\langle 0 | T\phi_{1I}(x_1) \dots \phi_{nI}(x_n) S | 0 \rangle}{\langle 0 | S | 0 \rangle},$$

and discussed how Greens functions are computed by summing all Feynman diagrams without vacuum bubbles.

Next topic: how to go from Green functions $\widetilde{G}^{(n)}(p_1, \dots, p_n)$, computed with external leg propagators, allowed to be off-shell, to S-matrix elements. E.g.

$$\langle k_3, k_4 | S - 1 | k_1 k_2 \rangle = \prod_{n=1}^{4} \frac{k_n^2 - m_n^2}{i} \widetilde{G}(-k_3, -k_4, k_1, k_2),$$

where the factors are to amputate the external legs. Consider for example $\widetilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4 \delta^{(4)}(k_1 + k_4) \frac{i}{k_1^2 - \mu^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(k_2 + k_3) \frac{i}{k_2^2 - \mu^2 + i\epsilon} + 2 \text{ permutations}.$$

This is the -1 that we subtract in $S - 1$, and indeed would not contribute to $2 \to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^{4}(k_n^2 - m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop.

• Introduce a source for $\phi(x)$, via $\delta\mathcal{L} = J(x)\phi(x)$. Then get diagrams where a meson ends on the source, with Feynman rule $i\widetilde{J}(k)$. At n-th order in J, the contribution to $\langle 0 | S | 0 \rangle$ is

$$\frac{i^n}{n!} \int \frac{d^4 k_1}{(2\pi)^3} \dots \int \frac{d^4 k_n}{(2\pi)^4} \widetilde{J}(-k_1) \dots \widetilde{J}(-k_n) \widetilde{G}^{(n)}(k_1, \dots, k_n)$$

Summing these up, we can write a functional of the source

$$Z[J] = \langle 0 | S | 0 \rangle_J = \langle 0 | T e^{i \int J(x)\phi(x)} | 0 \rangle.$$

Taking functional derivatives of it w.r.t. the source, using $\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$, gives the Green functions. Use Dyson's formula for $H = H_0 + H_1$ where H_0 is the full interacting Hamiltonian, and $\mathcal{H}_1 = -J(x)\phi_H(x)$, where now ϕ_H is in the Heisenberg picture. Then Dyson's formula gives

$$Z[J] = \langle\Omega|T\exp(i\int d^4x J(x)\phi_H(x))|\Omega\rangle,$$

which shows that $Z[J]$ is a generating functional for the Greens functions

$$G^{(n)}(x_1\ldots x_n) = Z[J]^{-1}\prod_{k=1}^{n}\frac{-i\delta}{\delta J(x_k)}Z[J]\big|_{J=0}.$$

• Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k\delta^{(3)}(\vec{k}'-\vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x}Z_\phi^{1/2}.$$

Now rescale the fields, s.t. $\langle k|\phi(x)|\Omega\rangle = e^{-ik\cdot x}$, and the LSZ formula is:

$$\langle q_1\ldots q_n|S-1|k_1\ldots k_m\rangle = \prod_{a=1}^{n}\frac{q_a^2-m_a^2}{i}\prod_{b=1}^{m}\frac{k_b^2-m_b^2}{i}\widetilde{G}^{(n+m)}(-q_1,\ldots -q_n,k_1,\ldots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.

To derive the LSZ formula, consider wave packets, with some profile $F(\vec{k})$, and $f(x) = \int\frac{d^3k}{(2\pi)^3 2\omega_k}F(\vec{k})e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2+\mu^2}$, so $f(x)$ solves the KG equation. Now define

$$\phi^f(t) = i\int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

Note that

$$i\int d^4x f(x)(\partial^2+\mu^2)\phi(x) = \int dt\phi^f(t).$$

$\phi^f(t)$ makes single particle wave packets from the vacuum, $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$. Also, $\langle\Omega|\phi^f(t)|k\rangle = 0$, and $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n}+p_n^0}{2\omega_{p_n}}F(\vec{p}_n)e^{-i(\omega_{p_n}-p_n^0)t}\langle n|\phi(0)\Omega\rangle$, where $\omega_{p_n}\equiv\sqrt{\vec{p}_n^2+\mu^2}$, which has $\omega_{p_n} < p_n^0$ for any multiparticle state. So $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. Then show

$$\langle f_m|S-1|f_n\rangle = \int \prod_n d^4x_n f_n(x_n) \prod_m d^4x_m f_m(x_m)^* \prod_r i(\partial_r^2 + m_r^2) G(x_n, x_m).$$

Take $f_i(x) \to e^{-ik_i x_i}$ at the end. To show the above, use that for arbitrary $\phi(x)$, and for KG solution $f(x)$, $i\int d^4x f(x)(\partial^2 + m^2)\phi(x) = (\lim_{t\to-\infty} - \lim_{t\to\infty})\phi^f(t)$. Show that all the $t \to \pm\infty$ do the right thing to give the in and out states, thanks to various cancellations, using $\lim_{t\to\pm\infty}\langle\Psi|\phi^f(t)|\Omega\rangle = \langle\Psi|f\rangle$.