10/28 Lecture outline

⋆ **Reading: Tong 3.7, Luke, chapter 8**

• Consider full interacting theory, with Hamiltonian H. Define the true vacuum $|\Omega\rangle$ such that $H|\Omega\rangle = 0$, and $\langle\Omega|\Omega\rangle = 1$.. The true vacuum of an interacting QFT is a complicated beast – it can be thought of roughly as a soup of particle-antiparticle states – it can not be solved for solved for exactly. (Progress: in classical mechanics, can solve 2 body problem exactly, but ≥ 3 body only approximately; in GR, can solve 1 body problem exactly, but ≥ 2 body only approximately; in QM can generally solve even only 1-body problem only approximately, but at least the 0-body problem is trivial; in QFT, even the 0-body problem is not exactly solvable.)

Define the Green functions or correlation functions by

$$G^{(n)}(x_1, \ldots x_n) = \langle\Omega|T\phi_H(x_1)\ldots\phi_H(x_n)|\Omega\rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian.

Now show that

$$G^{(n)}(x_1 \ldots x_n) = \frac{\langle 0|T\phi_{1I}(x_1)\ldots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

where $|0\rangle$ is the vacuum of the free theory, and ϕ_{iI} are interaction picture fields. To show this, take $t_1 > t_2 \ldots > t_n$ and put in factors of $U_I(t_a, t_b) = T\exp(-i\int_{t_a}^{t_b} H_I)$ to convert ϕ_I to ϕ_H, using $\phi_H(x_i) = U_I(t_i, 0)^\dagger \phi_I(x_i) U_I(t_i, 0)$. Get $\langle 0|U_I(\infty, t_1)\phi_H(t_1)\ldots\phi_H(t_n)U_I(t_n, -\infty)|0\rangle$, and U_I at ends can be replaced with full $U(t_1, t_2)$, since $H_0|0\rangle = 0$ anyway. Now use

$$\begin{aligned}\langle\Psi|U(t,-\infty)|0\rangle &= \langle\Psi|U(t,-\infty)\left(|\Omega\rangle\langle\Omega| + \sum\int |n\rangle\langle n|\right)|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\sum\int e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle\end{aligned}$$

where 1 was inserted as a complete set of states, including the vacuum and single and multiparticle states, including integrating over their momenta, but the wildly oscillating factor kills all those terms. (Riemann-Lebesgue lemma: $\lim_{t\to\infty}\int d\omega f(\omega)e^{i\omega t} = 0$ for nice $f(\omega)$) The result follows upon doing the same for the denominator.

The $\langle 0|S|0\rangle$ in the denominator eliminates the vacuum bubble diagrams. So we have

$$G^{(n)}(x_1, \ldots x_n) = \sum \text{Feynman graphs without vacuum bubbles.}$$

• Example: $G^{(4)}(x_1,x_2,x_3,x_4)$ in $\lambda\phi^4/4!$ theory. For each line from x to y, get a factor of $\Delta_F(x-y)$, and for each vertex at y get $-i\lambda\int d^4y$.

• It's more convenient often to work in momentum space,

$$\widetilde{G}^{(n)}(p_1,\dots p_n)=\int\prod_{i=1}^{n}d^4x_ie^{-ip_ix_i}G^{(n)}(x_1\dots x_n).$$

Similar to what we computed before to get S-matrix elements, but the external legs include their propagators, and the external momenta are not on-shell.

• From Green functions $\widetilde{G}^{(n)}(p_1,\dots,p_n)$, computed with external leg propagators, allowed to be off-shell, to S-matrix elements. E.g.

$$\langle k_3,k_4|S-1|k_1k_2\rangle=\prod_{n=1}^{4}\frac{k_n^2-m_n^2}{i}\widetilde{G}(-k_3,-k_4,k_1,k_2),$$

where the factors are to amputate the external legs. Consider for example $\widetilde{G}^{(4)}(k_1,k_2,k_3,k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon}+2\text{ permutations.}$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2\to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^4(k_n^2-m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop.