11/27 Lecture outline

Onward to statistics!

• Combinatoric factors count the number of configurations of a different type. For example, N distinguishable objects can be ordered $N!$ ways. With N distinguishable objects, we can form a set containing N_1 of them (with $N_2 = N - N_1$ left out) in "N choose N_1" $= \begin{pmatrix} N \\ N_1 \end{pmatrix} \equiv N!/N_1!(N-N_1)!$ distinct ways (not ordering the N_1 objects).

• Binomial distribution: event with 2 possible outcomes, #1 with probability p and #2 with probability q. E.g. coin tossing, where $p = q = \frac{1}{2}$ if the coin is unbiased. Another standard example: random walk. Consider N such events, and write $N = N_1 + N_2$ with N_1 the number of them with outcome #1 and N_2 that with outcome #2. The probability that $N = N_1 + N_2$ for a given choice of N_1 (and corresponding $N_2 = N - N_1$) is

$$p(N_1) = \begin{pmatrix} N \\ N_1 \end{pmatrix} p^{N_1} q^{N_2},$$

where $\begin{pmatrix} N \\ N_1 \end{pmatrix} \equiv N!/N_1!(N-N_1)!$ are the binomial coefficients, which enter e.g. in

$$(p+q)^N = \sum_{N_1=0}^{N} \begin{pmatrix} N \\ N_1 \end{pmatrix} p^{N_1} q^{N-N_1}.$$

Indeed, this condition shows that the probabilities are correctly normalized:

$$\sum_{N_1=0}^{N} p(N_1) = 1.$$

Using a little trick, we can also compute

$$\overline{N_1} = \sum_{N_1=0}^{N} N_1 p(N_1) = p\frac{\partial}{\partial p}(p+q)^N = Np$$

and

$$\overline{N_1^2} = \sum_{N_1=0}^{N} N_1^2 p(N_1) = \left(p\frac{\partial}{\partial p}\right)^2 (p+q)^N = (\overline{N_1})^2 + Npq.$$

So $\overline{(\Delta N_1)^2} = Npq$. I.e. $(\Delta N_1)_{RMS} = \sqrt{Npq}$, and $(\Delta N_1)_{RMS}/\overline{N}_1 = \sqrt{\frac{q}{p}}\frac{1}{\sqrt{N}}$. Distribution is very sharply peaked around $\overline{N}_1$ for large N.

• For very large N, use Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \text{for } n \gg 1.$$

Use this to approximate $\left(\begin{array}{c} N \\ N_1 \end{array}\right)$ when N and N_1 are both large. The above $p(N_1)$ then becomes, upon defining $x = N_1/N$ and $p(N_1)dN_1 = p(x)dx = p(x)dN_1/N$,

$$p(x) \to \frac{1}{\sqrt{2\pi}\sigma} \exp((x-\overline{x})^2/2\sigma^2) \qquad \text{with} \quad \overline{x} = p \quad \text{and} \quad \sigma = \sqrt{\frac{pq}{N}},$$

i.e. the Gaussian distribution. Height $\sim 1/\sigma \sim \sqrt{N}$, width $\sim \sigma \sim 1/\sqrt{N}$. For $N \to \infty$, the distribution is very sharply peaked around the average, $p(x) \to \delta(x - \overline{x})$.

• Omit in class, but if you're interested here are the details of how to get the gaussian via Stirling's approximation (along with a Taylor's series approximation). Write $\ln \left(\begin{array}{c} N \\ N_1 \end{array}\right) = \ln N! - \ln(Nx)! - \ln(N - Nx)!$. Using Stirling for each of the 3 terms, we have

$$\begin{aligned} \ln \left(\begin{array}{c} N \\ N_1 \end{array}\right) &\approx N \ln N - N + \tfrac{1}{2} \ln N + \tfrac{1}{2} \ln(2\pi) \\ &- [Nx \ln(Nx) + Nx + \tfrac{1}{2} \ln(Nx) + \tfrac{1}{2} \ln(2\pi)] \\ &- [N(1-x) \ln(N(1-x)) + \tfrac{1}{2} \ln(N(1-x)) + \tfrac{1}{2} \ln(2\pi)]. \end{aligned}$$

Expand this out and collect the terms. This function is peaked at $x = 1/2$, so Taylor expand it in x, around $x = 1/2$, and keep just the lowest order term involving x:

$$\ln \left(\begin{array}{c} N \\ N_1 \end{array}\right) \approx N \ln 2 - \tfrac{1}{2} \ln N - \tfrac{1}{2} \ln(\pi/2) - 2N(x - \tfrac{1}{2})^2 + O(x - \tfrac{1}{2})^4,$$

where the last term means order $(x - \frac{1}{2})^4$ and higher, and we now drop those terms, because their coefficients are all tiny (i.e. the function is sharply peaked). Exponentiating the above then gives

$$\left(\begin{array}{c} N \\ N_1 \end{array}\right) \approx 2^N \sqrt{\frac{2}{\pi N}} \exp(-2N(x - \tfrac{1}{2})^2).$$

This will give the quoted gaussian for the case $p = q = \frac{1}{2}$. For general p and q, when we multiply this by $p^{Nx} q^{N(1-x)}$, we get a function that is instead peaked at $x = p$. We should then Taylor expand $\ln \left(\begin{array}{c} N \\ N_1 \end{array}\right)$ instead around $x = p$. Doing that, and multiplying by $p^{Nx} q^{N(1-x)}$, gives the gaussian quoted above.

The binomial distribution, for large N can instead yield the Poisson distribution. It yields Gaussian if $N \gg 1$ and p is not going to zero, so $\overline{N_1} = Np$ is large. It yields Poisson if $N \gg 1$ and $p \to 0$, so that $Np \equiv a$ is held fixed. In that case, one expands around finite $N_1 \ll N$ and gets $P(N_1) \to a^{N_1} e^{-a}/N_1!$.

• Random walk (in 1 dimension): N steps, write as steps forward and steps backward, $N = N_+ + N_-$. Each step of length L. Distance traveled is $x = L(N_+ - N_-) = L(2N_+ - N)$. For each step, probabilities $p_+ + p_- = 1$. Probability of a given N_+ is given by binomial distribution. So $\overline{x} = L(2\overline{N}_+ - N) = L(2p-1)N$ and $\overline{(x^2)} = L^2(4\overline{N_+^2} - 4N\overline{N_+} + N^2)$, etc.

• Multi-nomial distribution, for where there are n outcomes possible. Binomial is $n = 2$, useful for coin tosses. $n = 4$ is useful for dreidels, and $n = 6$ is useful for dice. Fix $N = \sum_{i=1}^n N_i$; probability of a given set $\{N_i\}$ is

$$p(\{N_i\}) = N! \prod_{i=1}^n \frac{p_i^{N_i}}{N_i!},$$

where $\sum_{i=1}^n p_i = 1$. The number of states with $N = N_1 + \ldots N_n$ is $\omega(\{N_i\}) = N!/N_1! \ldots N_n!$. The total number of states is $\Omega = \sum'_{\{N_i\}} \omega(\{N_i\}) = n^N$. The $'$ on the sum means to sum over all values of the N_i, subject to the constraint that $\sum_{i=1}^n N_i = N$.

If the n outcomes are all equally, we have $p_i = 1/n$ and then

$$p(\{N_i\}) = \frac{\omega(\{N_i\})}{\Omega}, \qquad \omega(\{N_i\}) = N! \prod_{i=1}^n \frac{1}{N_i!},$$