11/7 Lecture outline

• Now let's figure out what $F(\vec{v})$ is. Note $F(\vec{v}) = F(v) = f(v_x)f(v_y)f(v_z)$, with $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. Take ∂_{v_x} ln of both sides. Conclude that $F(v)$ is of Gaussian normal form:

$$F(\vec{v}) = \left(\frac{\alpha}{\pi}\right)^{3/2} \exp(-\alpha \vec{v}^2),$$

where α is a constant, and the normalization factor ensures $\int F(\vec{v}) d^3\vec{v} = 1$. With this distribution, we easily compute $\overline{v^2} = \frac{3}{2}\alpha$ - this sets the size of the standard deviation of the probability distribution. To get our desired relation, $\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$, we see that we need the probability distribution to have $\alpha = m/2kT$, i.e.

$$F(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp(-\tfrac{1}{2}m\vec{v}^2/kT),$$

This is the Maxwell-Boltzmann velocity distribution. It is sharply peaked around $\vec{v}$ when T is small, and becomes a very broad distribution when T is large. This fits with our intuition: larger T means more jiggling of the molecules.

• Review gaussian distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x-\overline{x})^2/2\sigma^2),$$

where $\overline{x}$ is the mean and σ is the standard deviation. This distribution is common when there are large numbers in the sample. Note that

$$\int_{-\infty}^{\infty} (x-\overline{x})^n p(x) = \begin{cases} \frac{1}{\sqrt{\pi}} 2^{n/2}\sigma^n \Gamma(\frac{1}{2}(1+n)) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \qquad (1)$$

Here $\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}dt$ is the gamma function. By an integration by parts (with $u = t^z$ and $dv = e^{-t}dt$), you can show the gamma function satisfies the interesting property: $\Gamma(z+1) = z\Gamma(z)$. From this, it follows that $\Gamma(n) = (n-1)!$ for integer n, so the gamma function is sometimes called the factorial function. Also, find $\Gamma(1/2) = \sqrt{\pi}$ (and then $\Gamma(z+1) = z\Gamma(z)$ gives e.g. $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$). The above Eqn. (1) follows upon setting $t = (x-\overline{x})^2/2\sigma^2$ for n even (the integral clearly vanishes for n odd, since it's then an odd function of $\Delta x \equiv x - \overline{x}$, integrated over a range symmetric around $\Delta x = 0$).

In particular, the $n = 0$ case of Eqn. (1) gives $\int_{-\infty}^{\infty} p(x)dx = 1$, so $p(x)$ is correctly normalized. The $n = 1$ case of Eqn. (1) shows that $\int_{-\infty}^{\infty} xp(x) = \overline{x}$, so the $\overline{x}$ in $p(x)$ is indeed the mean value of x. The $n = 2$ case of (1) gives, upon defining $\Delta x \equiv x - \overline{x}$,

$\overline{\Delta x^2} = \overline{x^2} - (\overline{x})^2 = \sigma^2$. This shows how σ, which is the "standard deviation" sets the width of the gaussian distribution. We can write $\Delta x_{RMS} \equiv \sqrt{\overline{\Delta x^2}} = \sigma$.

• The Maxwell-Boltzmann velocity distribution

$$F(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp(-\tfrac{1}{2}m\vec{v}^2/kT),$$

is a product of gaussian distributions $F(\vec{v}) = p(v_x)p(v_y)p(v_z)$, each with zero mean, $\overline{v_x} = \overline{v_y} = \overline{v_z} = 0$ and standard deviation $\sigma = \sqrt{kT/m}$, so $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = kT/m$, which is the average energy equi-partition statement, $\overline{\frac{1}{2}mv_x^2} = \frac{1}{2}kT$ etc. So $v_{RMS} = \sqrt{\overline{v^2} - \overline{v}^2} = \sqrt{3kT/m}$. Also mean speed $\overline{v} = \int_0^\infty v(4\pi v^2)F(v)dv = \sqrt{8kT/\pi m}$. Most probable speed: $F(v)4\pi v^2$ is a maximum at $f(v)$ is a maximum at $v_{m.p.} = \sqrt{2kT/m}$.

• Write the Maxwell-Boltzman velocity distribution as an energy distribution: define $\epsilon \equiv \frac{1}{2}mv^2$ and define $p(\epsilon)d\epsilon = p(v)dv = 4\pi F(v)v^2dv$. Using $d\epsilon = mvdv$ and our expression for $F(v)$, and then writing v in terms of ϵ, this gives the energy distribution

$$p(\epsilon)d\epsilon = 2\pi^{-1/2}(kT)^{-3/2}\exp(-\epsilon/kT)\epsilon^{1/2}d\epsilon$$

which is the fraction of particles with energy in the range from ϵ to $\epsilon + d\epsilon$. It is properly normalized, as $\int_0^\infty p(\epsilon)d\epsilon = 1$.