$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2}(-\hat{L}^2)\Phi$$

$$\boxed{\text{note } \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)}$$

$$-\hat{L}^2\Phi = \frac{1}{\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \qquad \text{e.g. } \hat{L}_z = -i\frac{\partial}{\partial \phi}$$

$\hat{L}_z$ rotation generator around z axis, likewise for L_x & L_y

We know these e.g. from Q.M.: eigenvectors of $\hat{L}^2$ & $\hat{L}_z$ are $|l,m\rangle$

$$[\hat{L}_a, \hat{L}_b] = i\,\varepsilon_{abc}\hat{L}_c$$

$$\hat{L}^2|l,m\rangle = l(l+1)|l,m\rangle \qquad \hat{L}_z|l,m\rangle = m|l,m\rangle$$

$$Y_{lm}(\theta,\phi) \equiv \langle\theta,\phi|l,m\rangle \equiv \langle\Omega|l,m\rangle$$

spherical harmonics. $\Phi = \frac{U(r)}{r}F(\theta,\phi)$

Take $F(\theta,\phi) = \langle \theta,\phi | l, m\rangle \equiv Y_l^m(\theta,\phi)$

"spheical hamonics"

satisfies $\left(\frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)\right)$

$= -l(l+1)\,F$

since $\langle\theta,\phi| L^2 | l, m\rangle = \hbar^2 l(l+1)\langle\theta,\phi| l, m\rangle$

$$0 = \frac{Y_l^m}{r}u'' - \frac{u}{r^3}l(l+1)\,Y_l^m$$

so $u'' = \frac{1}{r^2}l(l+1)\,u$

$$u(r) = A r^{(l+1)} + B r^{-l}$$

$\boxed{m = \text{integer}}$

$\boxed{l = \text{integer}}$ $\boxed{l \geq 0}$

$\boxed{m = -l, \cdots l}$

as usual for QM

Using $-i\frac{\partial}{\partial\phi}Y_{lm}(\theta,\phi) = m\,Y_{l,m}(\theta,\phi)$

$$Y_{l,m}(\theta,\phi) \propto e^{im\phi}$$

$$Y_\ell^m(\theta,\phi) \sim P_\ell^m(x)\, e^{im\phi} \qquad x \equiv \cos\theta$$

$P_\ell^m(\cos\theta)$ satisfies $\quad x \equiv \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0$$

"generalized Legendre eqn"

for $m=0$ ordinary Legendre eqn

$P_\ell^{m=0} \equiv P(x)$ Legendre Polynomials

$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1)$

Useful generating function

$$F(h,x) = \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

$$\ell(\ell+1) \longrightarrow \frac{\partial}{\partial h} h^2 \frac{\partial}{\partial h}$$

$$0 = \frac{\partial}{\partial x}(1-x^2)\frac{\partial F}{\partial x} + \frac{\partial}{\partial h} h^2 \frac{\partial F}{\partial h}$$

Can use ~~~~ generating fn to $\to$ recursion rel'n

e.g. $$(n+1)P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0$$

Via $$(1-2hx+h^2)\frac{\partial F}{\partial h} = (x-h)F$$

Note $$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}$$

$|\vec{x}| = r$ suppose $r > r'$

$|\vec{x}'| = r'$

Legendre poly gen fn!

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\,\frac{1}{\sqrt{1-2\frac{r'}{r}\cos\theta+\frac{r'^2}{r^2}}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

ignore $\delta(\,)$

Of form of general sol'n of $\nabla^2\phi = 0$

$$\Phi = \sum_{l,m}\left(A_{l,m}\,r^l + \frac{B_{l,m}}{r^{l+1}}\right) Y_{l}^{m}(\theta,\phi)$$

for $m=0$ azimuthal symm $Y_l^m \to P_l(\cos\theta)$

Note from $\frac{1}{\sqrt{1-2hx+h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$

take $x=1$ $\quad \frac{1}{1-h} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x=1)$

$\Rightarrow$ all $P_\ell(x=1) = 1$

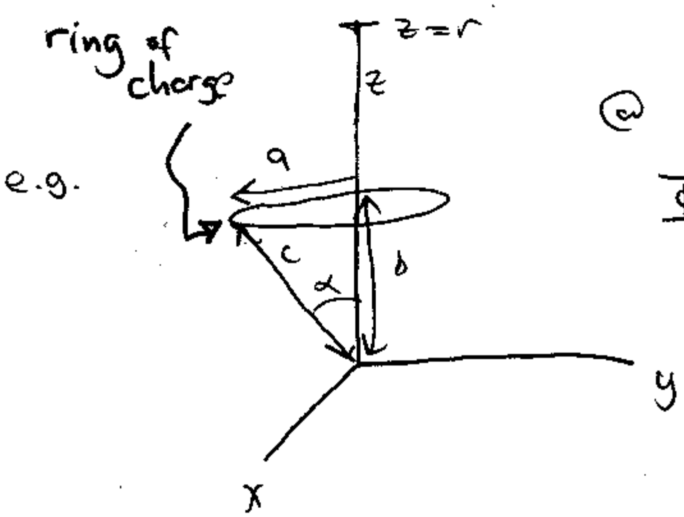
Solve for $\theta = 0$ $(\cos\theta = 1)$ & get general θ dep by putting in correct P_ℓ dep.

e.g. For azimuthal symm

$$\Phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$\Phi(r, \theta=0) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right)$$

e.g.



@ $x=y=0,\ z=r$

$$\Phi = \frac{q/4\pi\varepsilon_0}{(r^2+c^2-2rc\cos\alpha)^{1/2}}$$

So $\Phi(z=r) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r_<^{l}}{r_>^{l+1}} P_l(\cos\alpha)$

$r_<$ = smaller of r, c

$r_>$ = larger of r, c

$$\therefore \quad \Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r_<^{l}}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$$

is the sol'n for any point r, θ, ϕ in space, not necess on z axis. Magic!

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{n,m}$$

$$Y_{l,0} = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta) \leftarrow \text{normalized to } \delta_{l,l'}$$

More generally properly normalized $Y_{lm}(\theta,\phi)$

$= \langle \theta, \phi | l, m \rangle$ s.t. $\langle l' m' | l m \rangle = \delta_{ll'} \delta_{mm'}$

i.e. $\int d\Omega\, Y^*_{l'm'}(\Omega) Y_{lm}(\Omega) = \delta_{ll'} \delta_{mm'}$

likewise $1 = \sum_{l} \sum_{m=-l}^{l} | l m \rangle \langle l, m |$

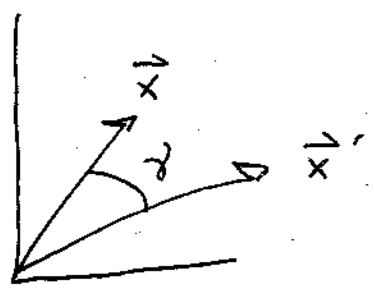
i.e. $\sum_{l} \sum_{m=-l}^{l} Y^*_{lm}(\Omega') Y_{lm}(\Omega) = \delta(\Omega - \Omega')$

Any $f(\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} B_{\ell m} Y_{\ell m}(\Omega)$

with $B_{\ell m} = \int d\Omega \, Y^*_{\ell m}(\Omega) f(\Omega)$.

$$\sum_{\ell m} Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega) = \delta(\Omega - \Omega')$$

$$\delta(\Omega - \Omega') = \sum_{\ell} B_\ell \, P_\ell(\cos\gamma)$$



$$\cos\gamma = \hat{x} \cdot \hat{x}' = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

$$B_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} d(\cos\gamma) \, \delta(\Omega - \Omega') \, P_\ell(\cos\gamma)$$

$$= \frac{2\ell+1}{4\pi} \int d\Omega \, \delta(\Omega - \Omega') \, P_\ell(\cos\gamma)$$

$$= \frac{2\ell+1}{4\pi} P_\ell(1) = \frac{2\ell+1}{4\pi}$$

$$\Rightarrow \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\cos\gamma) = \sum_{\ell m} Y^*_{\ell m}(\Omega') Y_{\ell m}(\Omega)$$

$$\Rightarrow \quad P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

"Addition thm" for spherical harmonics

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

$$= 4\pi \sum_{\ell, m} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

Recall Green fn $G(\vec{x},\vec{x}')$ s.t. $G\Big|_{|\vec{x}|=a} = 0$

obtained via image charge

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a/x'}{|\vec{x} - \frac{a^2}{x'^2}\vec{x}'|}$$

$$= 4\pi \sum_{\ell, m} \frac{1}{2\ell+1} \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1} \right) Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

eg. $r<r'$ $\quad \cancel{An}(\ ) = \frac{1}{r'^{\ell+1}} \left(\overbrace{r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}}^{\text{correct 2 terms in } r} \right)$