get $\int \frac{\vec{E}^2}{8\pi} d^3x = \sum_{a,b} \frac{q_a q_b}{2|\vec{x}_a - \vec{x}_b|}$

should take $a \neq b$ to avoid getting ∞.

e.g. for 2 particles finite part $= q_1 q_2 / |\vec{x}_1 - \vec{x}_2|$
Usual electostatic energy. ✓

Part II : solving more complicated cases

Jackson sometimes uses MKSA units.

Write $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ ← not $\frac{\vec{v}}{c}$ as before

note now $\vec{E}$ & $\vec{B}$ no longer have same units - bad for relativity, generally the transformations of $\vec{E}$ & $\vec{B}$ via relativity look more complicated in MKSA units.

Compared to before, scale $q_g \to k_Q q_{MKS}$
$\vec{E}_g \to k_Q^{-1} \vec{E}_{MKS}$ $\vec{B}_g \to c\, k_Q^{-1} \vec{B}_{MKS}$

old $\nabla \cdot \vec{E}_g = 4\pi\rho \Rightarrow k_Q^{-1} \nabla \cdot \vec{E}_{MKS} = 4\pi k_Q \rho_{MKS}$

Old $\nabla \times \vec{B}_g - \frac{1}{c}\frac{\partial \vec{E}_g}{\partial t} = \frac{4\pi}{c}\vec{J}_g$

$\Rightarrow \quad c\, k_Q^{-1}\, \nabla \times \vec{B}_{MKS} - \frac{1}{c} k_Q^{-1} \frac{\partial \vec{E}_{MKS}}{\partial t} = \frac{4\pi}{c} k_Q \vec{J}_{MKS}$

Old $\vec{\nabla} \times \vec{E}_g + \frac{1}{c}\frac{\partial \vec{B}_g}{\partial t} = 0 \Rightarrow \vec{\nabla} \times \vec{E}_{mks} + \frac{\partial \vec{B}_{mks}}{\partial t} = 0$

Take $k_Q^2 = 1/4\pi\varepsilon_0$

$\Rightarrow \quad \nabla \cdot \vec{E}_{MKS} = \rho/\varepsilon_0 \qquad \nabla \cdot \vec{B}_{MKS} = 0$

$\mu_0^{-1} \nabla \times \vec{B}_{MKS} - \varepsilon_0 \frac{\partial \vec{E}_{MKS}}{\partial t} = \vec{J}_{MKS}, \quad \mu_0^{-1} \equiv c^2 \varepsilon_0$

$\nabla \times \vec{E}_{MKS} + \frac{\partial \vec{B}_{MKS}}{\partial t} = 0.$

In Macroscopic media get:

$$\boxed{\begin{array}{cc} \vec{\nabla} \cdot \vec{D} = \rho & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} & \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

in vacuum $\vec{D} = \varepsilon_0 \vec{E} \quad \vec{B} = \mu_0 \vec{H}$

More generally $\vec{E} = \langle \vec{E}_{micro} \rangle \quad \vec{B} = \langle \vec{B}_{micro} \rangle$

But $\vec{D} = \vec{D}[\vec{E}, \vec{B}]$, $\vec{H} = \vec{H}[\vec{E}, \vec{B}]$ } complicated & possibly history dep. (hysteresis)

$$\frac{\partial D_i}{\partial E_j} = \varepsilon_{ij}$$ electric permittivity or dielectric tensor

$$\frac{\partial B_i}{\partial H_j} = \mu_{ij}$$ magnetic permeability tensor

$\varepsilon_{ij}[\vec{E}, \vec{B}]$ $\quad$ $\mu_{ij}[\vec{E}, \vec{B}]$ generally complicated

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Electrostatic system of stationary charges

point charges q_i @ positions $\vec{x}_i$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$ superposition

$$\vec{E} = -\nabla\phi, \qquad \phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|}$$

$$\rho(\vec{x}) = \sum_i q_i\, \delta^3(\vec{x} - \vec{x}_i)$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow -\nabla^2\phi = \rho/\varepsilon_0$$

so $$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}_i|}\right) = -4\pi\,\delta^3(\vec{x}-\vec{x}_i)$$

Many ways to show (e.g. Fourier transform)

here just note in spherical coords

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$
$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

so formally $\nabla^2\left(\frac{1}{r}\right) = 0$, valid everywhere except at $r=0$. Gauss law

$$\int_V \nabla^2\left(\frac{1}{r}\right) = \int_S \vec{\nabla}\left(\frac{1}{r}\right)\cdot d\vec{S} = \int_S -\frac{1}{r^2}\hat{r}\cdot d\vec{S}$$

$d\vec{S} = \hat{r}\,r^2\underbrace{d(\cos\theta)\,d\phi}_{d\Omega}$ so $\int_V \nabla^2\left(\frac{1}{r}\right) = -\int_S d\Omega$

$= -4\pi$ if $r=0$ inside S or zero otherwise

$$\nabla^2\left(\frac{-1}{4\pi|\vec{x}-\vec{x}_i|}\right) = \delta^3(\vec{x}-\vec{x}_i)$$

"Green function of Laplacian ∇^2"

Solve $\nabla^2\phi = -\rho/\varepsilon_0$ via "Poisson Eqn"

$$\phi(\vec{x}) = \int d^3\vec{y}\ \frac{1}{4\pi\varepsilon_0}\ \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

$$\nabla^2\phi(\vec{x}) = \int d^3\vec{y}\ \frac{1}{4\pi\varepsilon_0}\ \rho(\vec{y})\ \left(-4\pi\,\delta^3(\vec{x}-\vec{y})\right)$$

$$= -\rho(\vec{x})/\varepsilon_0\ \checkmark$$

Plug in $\rho(\vec{y}) = \sum_i q_i\ \delta^3(\vec{y}-\vec{x}_i)$

get $\phi(\vec{x}) = \sum_i \frac{q_i}{4\pi\varepsilon_0}\ \frac{1}{|\vec{x}-\vec{x}_i|}$ as before

More general Green function:

$$\nabla_x^2\ G(\vec{x},\vec{y}) = -4\pi\,\delta^3(\vec{x}-\vec{y})$$

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} + F(\vec{x},\vec{y})$$

$\nabla^2 F(\vec{x},\vec{y}) = 0$ "Laplace eqn"

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3\vec{y}\ \ G(\vec{x},\vec{y})\,\rho(\vec{y})$$

$$+\frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\ \nabla'\phi - \phi(\vec{x}')\,\nabla' G(\vec{x},\vec{x}')\right]\cdot\hat{n}\,da'$$

Obtain from math identity

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d^3x = \oint_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot d\vec{A}$$

for $\psi = \frac{1}{4\pi\varepsilon_0} G$.

Specify F & $\therefore$ G by choice of BCs eg

"Dirichlet" specify $\Phi|_{bndy}$

"Neumann" " $\vec{n}\cdot\nabla\Phi|_{bndy}$

e.g. Dirichlet $\Rightarrow$ $G(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on bndy

Electrostatic energy: in CGS we had $H_{field} = \int \frac{d^3x}{8\pi} (\vec{E}^2 + \vec{B}^2)$

now $\vec{B} = 0$ here & $\frac{\vec{E}^2_{CGS}}{8\pi} \longrightarrow \frac{\varepsilon_0}{2} \vec{E}^2_{MKS}$

so $H_{field} = \frac{\varepsilon_0}{2} \int d^3x \, \vec{E}^2$ (for $\vec{B} = 0$)

$\vec{E} = -\nabla\phi \hookrightarrow H_{field} = \frac{\varepsilon_0}{2} \int d^3x \, \nabla\phi \cdot \nabla\phi$

$$H_{field} = \frac{\varepsilon_0}{2} \int d^3x \left[\nabla(\phi \nabla\phi) - \phi \nabla^2 \phi \right]$$

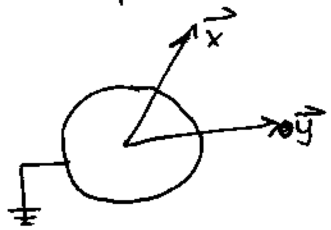
$$= \frac{\varepsilon_0}{2} \oint_S \phi \nabla\phi \cdot d\vec{S} - \frac{\varepsilon_0}{2} \int d^3x \, \phi \left(-\frac{\rho(\vec{x})}{\varepsilon_0} \right)$$

$$= \text{surface contrib} + \frac{1}{8\pi\varepsilon_0} \iint \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x \, d^3x'$$

must exclude self energy at $\vec{x} = \vec{x}'$ to get finite answer. E.g. $\rho = \sum_i q_i \delta(\vec{x} - \vec{x}_i)$

$$H_{field} = \text{surface term} + \frac{1}{8\pi\varepsilon_0} \sum_{i,j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

Example of B.C. problem



Ground conducting sphere charge q @ $\vec{y}$, observer @ $\vec{x}$

$$\phi(\vec{x}) = \frac{q/4\pi\varepsilon_0}{|\vec{x}-\vec{y}|} + \frac{q'/4\pi\varepsilon_0}{|\vec{x}-\vec{y}'|}$$ ← image charge contrib.

$$\vec{y}' = y'\,\hat{n}' \qquad \vec{y} = y\,\hat{n}' \qquad \vec{x} = x\,\hat{n}$$

$$\phi(\vec{x})\Big|_{|\vec{x}|=a} = 0 \qquad \leftarrow \text{B.C.}$$

$$\phi(a) = \frac{q/4\pi\varepsilon_0}{|a\hat{n} - y\hat{n}'|} + \frac{q'/4\pi\varepsilon_0}{|a\hat{n} - y'\hat{n}'|} = 0$$

$$\frac{q/4\pi\varepsilon_0}{a\,|\hat{n} - \frac{y}{a}\hat{n}'|} + \frac{q'/4\pi\varepsilon_0}{y'\,|\hat{n}' - \frac{a}{y'}\hat{n}|} = 0$$

sol'n: $\quad q/a = -q'/y' \qquad \frac{y}{a} = \frac{a}{y'}$

$$\Rightarrow \quad q' = -aq/y \qquad y' = a^2/y$$

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2-2xy\cos\gamma}} + \frac{-a/y}{\sqrt{x^2+\frac{a^4}{y^2}-2\frac{xa^2}{y}\cos\gamma}}\right)$$

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left(\left(x^2+y^2-2xy\cos\gamma\right)^{-1/2} - \left(\frac{x^2y^2}{a^2}+a^2-2xy\cos\gamma\right)^{-1/2}\right)$$

$x = |\vec{x}|$ $\quad y = |\vec{y}|$ here, of course

Can find surface charge

$$\sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial x}\right|_{x=a}$$

$$\sigma = \frac{+q}{4\pi}\underbrace{\left(a^2+y^2-2ay\cos\gamma\right)}_{*}^{-3/2}\left(a - y\cancel{\cos\gamma} - \frac{1}{a}y^2 + y\cancel{\cos\gamma}\right) = -\frac{aq}{4\pi}\left(y^2-a^2\right)\left(*\right)^{-3/2}$$

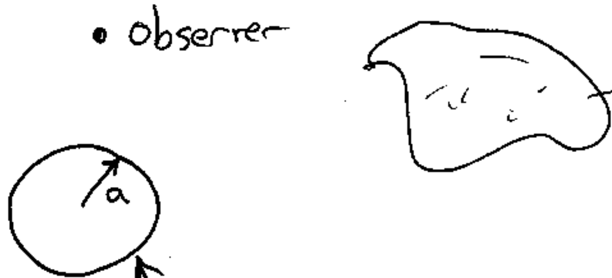
Can verify $\oint d\Omega\, a^2\sigma = -\frac{qa}{y} = q'$ ✓

To get insulated rather than ground conductor add charge $Q-q'$ smoothly distributed

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|} + \frac{\left(Q + \frac{aq}{y}\right)}{|\vec{x}|}\right]$$

Also fixed potential @ sphere $Q - q' \to Va$

Consider now complicated looking problem



charge distribution ρ in space

sphere with given potential $\phi(a,\theta,\phi)$

Solve using Green function

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$

$$= \left(x^2 + x'^2 - 2xx'\cos\gamma\right)^{-1/2} - \left(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{-1/2}$$

γ = angle between $\vec{x}$ & $\vec{x}'$

G=0 on sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, G(\vec{x},\vec{x}')\rho(\vec{x}') + \frac{1}{4\pi}\oint_S \left(G\nabla\Phi - \Phi\nabla G\right)\cdot d\vec{S}$$

So $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \quad G(\vec{x}, \vec{x}')\rho(\vec{x}')$

$$-\frac{1}{4\pi} \int d(\cos\theta')\, d\phi' \; \Phi(a, \theta', \phi') \; \left.\frac{\partial G}{\partial n'}\right|_{x'=a}$$

$$\frac{\partial G}{\partial n'} = - \frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax\cos\gamma)^{3/2}}$$

The problem is solved (assuming we can evaluate the integral)!

Study sol'ns of Laplace eqn $\nabla^2 \Phi = 0$

e.g. in box $\quad \Phi = X(x)\, Y(y)\, Z(z)$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

each must be constant $\quad X'' = c_x X$

$c_x + c_y + c_z = 0 \quad\quad Y'' = c_y Y$

$Z'' = c_z Z$

sol'ns $= \Sigma \underset{c<0}{\sin \,\&\, \cos}$. or $\Sigma \underset{c>0}{\sinh \,\&\, \cosh}$

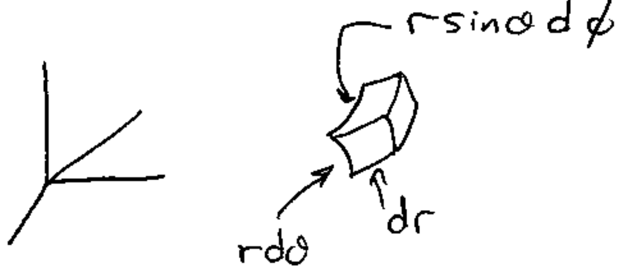
Spherical coordinates: $\nabla^2 \Phi$ ~~aaa~~ $=$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$\boxed{\begin{aligned} ds^2 &= \sum_n h_n^2\, d\xi_n^2 \\ \nabla f &= \sum_n \frac{\hat{n}}{h_n}\frac{\partial f}{\partial \xi_n} \\ \vec{\nabla}\cdot\vec{F} &= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial\xi_1}(F_1h_2h_3) + \cdots\right) \\ \vec{\nabla}\times\vec{F} &= \frac{\hat{a}_1}{h_2h_3}\left(\frac{\partial}{\partial\xi_2}(h_3F_3) - \frac{\partial}{\partial\xi_3}(h_2F_2)\right) + \cdots \end{aligned}}$$

aside:



$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta A_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta A_\theta\right)\right.$$

$$\left. + \frac{\partial}{\partial\phi}\left(rA_\phi\right)\right]$$

understand via Gauss law

explain

$$\vec{\nabla}\times\vec{A} = \frac{\hat{r}}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(r\sin\theta A_\phi\right) - \frac{\partial}{\partial\phi}\left(rA_\theta\right)\right)$$

$$+\frac{\hat{\theta}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(A_r) - \frac{\partial}{\partial r}\left(r\sin\theta A_\phi\right)\right) + \frac{\hat{\phi}}{r}(\cdots)$$