$$T_{\text{matter}}^{\mu\nu} = \frac{D}{\gamma} u^\mu u^\nu = \sum_a \frac{m_a}{\gamma} u^\mu u^\nu \delta^3(\vec{x}-\vec{x}_a)$$

$$= \sum_a \frac{p_a^\mu p_a^\nu}{E_a/c^2} \delta^3(\vec{x} - \vec{x}_a(t)) \quad \text{as before}$$

More general stress-energy tensor:

$$c^2 T^{\mu\nu} = (p+\varepsilon) u^\mu u^\nu - p g^{\mu\nu}$$

ε = energy/vol $\qquad$ p = pressure

in rest frame this $c^2 T^{\mu\nu} = \begin{pmatrix} \varepsilon & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$ ✓

Note $T_\mu{}^\mu$ = lorentz invt $= \frac{1}{c^2}(\varepsilon - 3p)$

can generally argue $T_\mu{}^\mu \geq 0$ so $\varepsilon \geq 3p$

$\varepsilon = 3p$ is "ultrarelativistic."

Can show via "virial thm" $\varepsilon - 3p = \sum_a m_a c^2 \sqrt{1 - \frac{v_a^2}{c^2}}$

$= 0$ for massless particles or v_a all $\to c$.

Follows from $T_{\text{matter}}^{\mu\nu} = \frac{D}{\gamma} u^\mu u^\nu \to T_{\text{matter}}{}_\mu{}^\mu = \frac{D}{\gamma} c^2$

From $T^{\mu\nu}_{matt} = \frac{D}{\gamma} U^\mu U^\nu = D^\mu U^\nu$

get $\partial_\mu T^{\mu\nu}_{matt} = \underbrace{\cancel{(\partial_\mu D^\mu)}}_{\substack{0 \\ \text{matt. cons.}}} U^\nu + D^\mu \partial_\mu U^\nu$

↳ $\partial_\mu T^{\mu\nu}_{matt} = D \frac{dx^\mu}{dt} \frac{\partial}{\partial x^\mu} U^\nu = D \frac{d}{dt} U^\nu$

e.o.m. : $m \frac{du^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} U_\nu$

↳ $m\gamma \frac{du^\mu}{dt} = \frac{q}{c} F^{\mu\nu} U_\nu$

replace $m \longrightarrow D$ matter density

$q \longrightarrow \rho$ charge density

Force eqn $\Rightarrow$ $D\gamma \frac{du^\mu}{dt} = \frac{\rho}{c} F^{\mu\nu} \frac{dx_\nu}{dt} \gamma$

$\Rightarrow D \frac{du^\mu}{dt} = \frac{1}{c} F^{\mu\nu} J_\nu$ $\left(J_\nu = \rho \frac{dx^\nu}{dt} \right)$

↑ charge, current density 4-vect

so $\boxed{\partial_\mu T^{\mu\nu}_{matt} = \frac{1}{c} F^{\nu\lambda} J_\lambda}$ ↖ matter energy mom. not cons.

$\partial_\mu T^{\mu\nu}_{matt} = \frac{1}{c} F^{\nu\lambda} J_\lambda \neq 0$

$\Rightarrow$ matter energy / momentum not conserved!

As seen in examples. Resolution: ~~[illegible]~~

must include field energy / momentum.

$\mathcal{L}_{total} = \mathcal{L}_{matter} + \mathcal{L}_{field} + \mathcal{L}_{int}$ has

x^μ translational inv $\Rightarrow$ $T^{\mu\nu}_{total}$ conserved

$\partial_\mu T^{\mu\nu}_{total} = 0$. Follows from Noether's thm

relating symmetries $\longleftrightarrow$ cons. laws. : e.g. field

$$\frac{d\mathcal{L}_{field}}{dx_\mu} = \frac{\partial \mathcal{L}_{field}}{\partial A_\lambda} \partial^\mu A_\lambda + \frac{\partial \mathcal{L}_{field}}{\partial(\partial_\nu A_\lambda)} \partial^\mu \partial_\nu A_\lambda + \frac{\partial \mathcal{L}_{field}}{\partial x_\mu}$$

plug in Euler Lagrange eqns: $\frac{\partial \mathcal{L}}{\partial A_\lambda} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)}\right)$

$$\Rightarrow \frac{d\mathcal{L}}{dx_\mu} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\lambda)} \partial^\mu A_\lambda\right) + \frac{\partial \mathcal{L}}{\partial x_\mu}$$

$$\Rightarrow \partial_\nu T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\mu}$$ $\leftarrow$ ~~with~~ extenal force $= 0$ if transl. inv't.

with $$T^{\mu\nu}_{\text{field}} \equiv \frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\nu A_\lambda)} \partial^\mu A_\lambda - g^{\mu\nu} \mathcal{L}_{\text{field}}$$

ambiguity: $T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda \psi^{\mu\nu\lambda}$

doesn't change $\underline{P}^\mu = \int d^3x\, T^{\mu 0}$

& doesn't change $\partial_\mu T^{\mu\nu}$ if $\psi^{\mu\nu\lambda}$ is antisymm in $\mu \leftrightarrow \lambda$. We should take $T^{\mu\nu} = T^{\nu\mu}$ by choice of $\psi^{\mu\nu\lambda}$.

Why need $T^{\mu\nu} = T^{\nu\mu}$: angular momentum cons.

$$M^{\mu\nu} = \frac{1}{c^2} \int d^3x \left(x^\mu T^{\nu 0} - x^\nu T^{\mu 0} \right)$$

like $\vec{r} \times \vec{p}$ density. Conserved if

$$\partial_\lambda \left(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda} \right) = 0$$

Using $\partial_\lambda T^{\mu\lambda} = 0$ this is true

iff $T^{\mu\nu} = T^{\nu\mu}$ symmetric stress tensor

so $T^{\mu\nu}_{total} \stackrel{!}{=}$ symmetric. Already found

$T^{\mu\nu}_{matter}$ = symmetric.

$$\mathcal{L}_{tot} = \mathcal{L}_{matt} + \mathcal{L}_{field} + \mathcal{L}_{int}$$

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{field} + \cancel{T^{\mu\nu}_{int}}$$

0 appropriately interpreting $T^{\mu\nu}_{field}$ - more soon.

$$\mathcal{L}_{field} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \qquad \frac{\partial \mathcal{L}_{field}}{\partial(\partial_\mu A_\lambda)} = -\frac{F^{\mu\lambda}}{4\pi}$$

$$\hookrightarrow \; T^{\mu\nu}_{field} = -\frac{F^{\mu\lambda}}{4\pi} \partial^\nu A_\lambda + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$+ \partial_\lambda \psi^{\mu\nu\lambda} \qquad \text{take } \psi^{\mu\nu\lambda} = \frac{1}{4\pi} F^{\mu\lambda} A^\nu$$

$$\boxed{T^{\mu\nu}_{field} = -\frac{1}{4\pi} F^{\mu\lambda} F^{\nu}{}_{\lambda} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$

note gauge invarient! good!

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{field}$$

$$\partial_\mu T^{\mu\nu}_{field} = \frac{1}{4\pi}\Big(-\partial_\mu F^{\mu\lambda} F^{\nu}{}_{\lambda} - \cancel{F^{\mu\lambda}\partial_\mu F^{\nu}{}_{\lambda}}$$

$$+ \frac{1}{2} \cancel{F^{\alpha\beta}\partial^\nu F_{\alpha\beta}} \Big) = -\frac{1}{c} J^{\lambda} F^{\nu}{}_{\lambda}$$

Where we used the Maxwell eqns $\partial_\mu F^{\mu\lambda} = \frac{4\pi}{c} J^{\lambda}$

and $\partial^\nu F_{\alpha\beta} + \partial_\alpha F_{\beta}{}^{\nu} + \partial_\beta F^{\nu}{}_{\alpha} = 0$

$\hookrightarrow$ $F^{\alpha\beta}\partial^\nu F_{\alpha\beta} + 2F^{\alpha\beta}\partial_\alpha F_{\beta}{}^{\nu} = 0$

Why above two terms cancel

$\hookrightarrow$ $\partial_\mu T^{\mu\nu}_{field} = -\frac{1}{c} F^{\nu\lambda} J_\lambda$

recall $\partial_\mu T^{\mu\nu}_{matter} = +\frac{1}{c} F^{\nu\lambda} J_\lambda$

so $\partial_\mu \left(T^{\mu\nu}_{field} + T^{\mu\nu}_{matter} \right) = 0$ conserved! ✓

Write out $T^{\mu\nu}_{field}$:

$$T^{00}_{field} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right) \equiv u \quad \text{field energy/density}$$

"Poynting vector"

$$T^{0i}_{field} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i \equiv \frac{S^i}{c} \quad c\cdot\text{momentum density}$$

$$T^{ij}_{field} = -\frac{1}{4\pi}\left(E_i E_j + B_i B_j - \frac{1}{2}\delta^{ij}\left(\vec{E}^2+\vec{B}^2\right)\right)$$

$$\mathcal{H}_{field} = \frac{1}{8\pi}\int d^3x \left(\vec{E}^2 + \vec{B}^2\right) \qquad \text{field energy}$$

$$c\underline{P}_{field} = \frac{1}{4\pi}\int d^3x\, \vec{E}\times\vec{B} = \int d^3x\, \vec{S}/c$$

$$\vec{L}_{field} = \frac{1}{4\pi c}\int d^3x \quad \vec{X}\times\vec{E}\times\vec{B} \qquad \text{field angular momentum}$$

In all of these $\vec{E}$ & $\vec{B}$ are the total field, including that of all particles. This is how $\mathcal{H}_{field} + \mathcal{H}_{matter}$, for example, end up properly including interaction energy associated with $\mathcal{L}_{int}$.

E.g. mass m charge q particle interacting with $\vec{E}_{ext}$ external field. Before we had

$$H = \gamma m c^2 + q\phi_{ext}$$

now we have $H_{matter} = \gamma m c^2$

$$H_{field} = \frac{1}{8\pi}\int d^3x \, \vec{E}^2$$

Q: Where is $q\phi_{ext}$ term?

A: $\vec{E} = \vec{E}_{ext} + \vec{E}_q$ ← electric field due to charge q particle

$$\vec{E}^2 = \vec{E}_{ext}^2 + \vec{E}_q^2 + 2\vec{E}_{ext}\cdot\vec{E}_q$$

oo oops! ← ignore!

$$H_{field} = \frac{1}{8\pi}\int d^3x\, \vec{E}_{ext}^{\,2} + \cancel{\frac{1}{8\pi}\int d^3x\, \vec{E}_q^{\,2}} + \frac{1}{4\pi}\int \vec{E}_{ext}\cdot\vec{E}_q\, d^3x$$

$$\vec{E}_{ext} = -\nabla\phi_{ext} \quad \text{so} \quad \frac{1}{4\pi}\int d^3x\,(-\nabla\phi_{ext})\cdot\vec{E}_q \quad \text{in } H_{field}$$

integrate by parts $\frac{1}{4\pi}\int d^3x\, \phi_{ext}\, \nabla\cdot\vec{E}_q$

$$\nabla\cdot\vec{E}_q = 4\pi\rho = 4\pi q\,\delta^3(\vec{x}-\vec{x}_0)$$

so get energy term $\phi_{ext}\, q$ as found earlier associated w/ interaction

Another example: $\rho = \sum_a q_a\,\delta^3(\vec{x}-\vec{x}_a)$

$$\vec{E} = \sum_a \vec{E}_a, \quad \vec{E}_a = -\nabla\phi_a, \quad \phi_a = \frac{q_a}{|\vec{x}-\vec{x}_a|}$$

$$\vec{E}^2 = \sum_{a,b}\vec{E}_a\cdot\vec{E}_b$$

$$\int\frac{\vec{E}^2}{8\pi}d^3x = \sum_{a,b}\int\frac{-\nabla\phi_a\cdot\vec{E}_b}{8\pi}d^3x = \frac{1}{2}\sum_{a,b}\int\phi_a\,\rho_b\,d^3x$$

$$\rho_b = q_b\,\delta(\vec{x}-\vec{x}_b)$$

get $\displaystyle\int \frac{\vec{E}^2}{8\pi}\, d^3x = \sum_{a,b} \frac{q_a q_b}{2|\vec{x}_a - \vec{x}_b|}$

should take $a \neq b$ to avoid getting ∞.

e.g. for 2 particles finite part $= q_1 q_2 / |\vec{x}_1 - \vec{x}_2|$

usual electrostatic energy. ✓