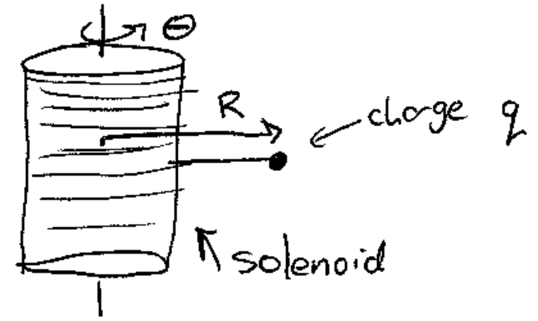
Another example: (from vol. 2 Feynman lects in physics Fig 17-5)



moment of inertia

Non-rel limit: $L = \frac{I}{2}\dot{\theta}^2 + \frac{q}{c} A_\theta(R) R\dot{\theta}$

$2\pi R\, A_\theta(R) = \oint \vec{A}\cdot d\vec{\ell} = \int \vec{\nabla}\times\vec{A}\cdot ds = B(\pi a^2) \equiv \Phi$

magnetic flux

rotational inv $\Rightarrow$ $P_\theta = I\dot{\theta} + \frac{q}{c}A_\theta R = \text{const}$

so $I\dot{\theta} + \frac{q}{c}\frac{\Phi}{2\pi} = \text{const.}$

Suppose initially $\dot{\theta} = 0$, $\Phi \neq 0$. Now heat up until current no longer flows in coil $\Phi \to 0$

$\therefore$ $\dot{\theta}_{final} = \frac{-q}{Ic}\frac{\Phi_{initial}}{2\pi} \neq 0$, starts to spin. What about conservation of $\vec{L}$? (No external torque!) Ans.: must take into account $\vec{L}$ of electric-magnetic field.

$\vec{E}$ & $\vec{B}$ in relativistic form: electro-magnetic field strength tensor

$$F_{\mu\nu} = -F_{\nu\mu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = g_{\mu\sigma} g_{\nu\rho} F^{\sigma\rho}$$

Under Lorentz transf $\quad X^\mu = \Lambda^\mu{}_{\nu'} X'^{\nu'}$

$$F^{\mu\nu} = \Lambda^\mu{}_{\sigma'} \Lambda^\nu{}_{\chi'} F'^{\sigma'\chi'}$$

antisymm in all ref. frames.

e.g. $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \qquad \begin{matrix} y = y' \\ z = z' \end{matrix}$

$$F^{01} = \gamma\left(F^{0'1} + \beta F^{1'1}\right) = \gamma^2\left(F^{0'1'} + \beta^2 F^{1'0'}\right) = F'^{0'1'}$$

$$F^{23} = F'^{2'3'}$$

$\hookrightarrow B_x = B_x'$ ✓

$\hookrightarrow E_x = E_x'$ ✓

$$\begin{pmatrix} -E_y \\ -B_z \end{pmatrix} \leftarrow \begin{pmatrix} F^{02} \\ F^{12} \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} F'^{02} \\ F'^{12} \end{pmatrix}$$

$$\begin{pmatrix} -E_z \\ B_y \end{pmatrix} \leftarrow \begin{pmatrix} F^{03} \\ F^{13} \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} F'^{03} \\ F'^{13} \end{pmatrix}$$

So we get correct Lorentz transf. laws found in earlier exercise!

Lorentz force law $\quad \frac{d}{d\tau}(m u^\mu) = f^\mu$

$$f^\mu = \frac{q}{c}\left(\vec{u}\cdot\vec{E},\ u_0\vec{E} + \vec{u}\times\vec{B}\right) \overset{*}{=} \frac{q}{c} F^{\mu\nu} u_\nu$$

$*$ = show this. $\qquad$ manifestly transfs. as 4-vector

Maxwell eqns: $\quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad \xrightarrow{*} \quad \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

4-vector of eqns!

$J^\nu \equiv (c\rho, \vec{J})$ ← 4 current density, transforms as 4-vector!

if $X^\mu = \Lambda^\mu{}_\nu X'^\nu$, then $J^\mu = \Lambda^\mu{}_\nu J'^\nu$

charge in vol d^3x: $\quad \rho\, d^3x = \rho'\, d^3x'$ (charge cons)

Compare to $\quad d^4x = \det\Lambda\, d^4x' = d^4x'$

$\Rightarrow$ ρ transforms as dt = time compt. of 4-vector under Lorentz transfs.

Take $\quad \rho = \sum_a q_a\, \delta^3(\vec{x} - \vec{x}_a(t))$

$$\vec{J} = \sum_a q_a \frac{d\vec{x}_a}{dt}\, \delta^3(\vec{x} - \vec{x}_a(t))$$

$$\therefore\ J^\mu \equiv (c\rho, \vec{J}) = c\rho\frac{dx^\mu}{dx^0} = \text{4-vector} \checkmark$$

Charge cons: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \leftrightarrow \boxed{\partial_\mu J^\mu = 0}$

Lorentz inv. eqn!

Maxwell eqns $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

automatic! $\hookrightarrow$ Charge cons. since $\partial_\nu J^\nu = \frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0$

since $\partial_\mu \partial_\nu$ is symm in $\mu \leftrightarrow \nu$ whereas $F^{\mu\nu}$ is antisymm.

Other 2 Maxwell eqns :
$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \end{array}\right] (2)$$

(2)''

Can be written as $\boxed{\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0}$

$\updownarrow$

or as $\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$ $\quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$\tilde{F}^{\mu\nu}$ similar to $F^{\mu\nu}$ but w/ $\vec{E} \to \vec{B}$ & $\vec{B} \to -\vec{E}$

Can solve (2) via $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$

get $\partial_\mu \partial_\rho \varepsilon^{\mu\nu\rho\sigma} A_\sigma = 0$ trivially since $\partial_\mu \partial_\rho$ symm in $\mu \leftrightarrow \rho$ & $\varepsilon^{\mu\nu\rho\sigma}$ is antisymm.

Writing $A^\mu = (\phi, \vec{A})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A} \ \checkmark$$

as usual

Plug $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ into

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \Rightarrow \Box A^\nu - \partial^\nu(\partial \cdot A) = \frac{4\pi}{c} J^\nu$$

w/ $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ ← Lorentz invt "d'alembertian"

$$\partial \cdot A \equiv \partial_\mu A^\mu = \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A}$$

Can choose "Lorentz gauge" $\partial_\mu A^\mu = 0$

↳ $\Box A^\nu = \frac{4\pi}{c} J^\nu$ as seen earlier.

Can get Maxwell's eqns as Euler-Lagrange eqns for action $S = \int dt\, L = \int \underbrace{d^4x}_{= dt\, d^3x} \mathcal{L}$

$\mathcal{L}$ = Lagrangian density

Analogy w/ electromagnetic field ~ fluid.

Go back to action for collection of free particles: $S_m = \sum_a \int -m_a c^2\, d\tau$

define $D \equiv \sum_a m_a\, \delta^3(\vec{x} - \vec{x}_a)$ ~ mass density

$$S_m = -c^2 \int D\, d^3\vec{x}\, d\tau = -c^2 \int \frac{D}{\gamma}\, d^4x$$

Let $D^\mu = D\frac{dx^\mu}{dt} = (cD, D\vec{v})$

transforms as 4-vector

analogous to $J^\mu = (c\rho, \rho\vec{v})$ charge/current density 4-vect.

Just as $\partial_\mu J^\mu = 0 \longleftrightarrow$ charge cons.

$\partial_\mu D^\mu = 0 \longleftrightarrow$ mass cons.

(free non-interacting particles!)

So $S_{matter} = \int d^4x \, \mathcal{L}_{matter}$ with $\boxed{\mathcal{L}_{matter} = -\frac{c^2 D}{\gamma}}$

i.e. $\mathcal{L}_{matter} = -c^2 D(x^\mu)\sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}}$

S action should be Lorentz invt & d^4x is Lorentz invt so Lagrangian densities $\mathcal{L}$ must be Lorentz invt. $\mathcal{L}_{matter}$ = Lorentz invt. ✓

Can also write $S_{int} = \sum_a -\frac{q_a}{c}\int A_\mu dx^\mu$

in terms of Lagrangian density:

$\rho = \sum_a q_a \, \delta^3(\vec{x} - \vec{x}_a(t))$, $J^\mu = c\rho\frac{dx^\mu}{dx^0}$

$S_{int} = \int d^4x \, \mathcal{L}_{int}$ w/ $\boxed{\mathcal{L}_{int} = -\frac{1}{c}A_\mu J^\mu}$