Note $S = \int \left(-mc\,ds - \frac{q}{c} A_\mu dx^\mu\right)$

is gauge invt. under $\vec{A} \to \vec{A} + \nabla f$

$$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

$A^\mu = (\phi, \vec{A})$ $\quad \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

so gauge transf is $A^\mu \to A^\mu - \partial^\mu f$

or write as $A_\mu \to A_\mu - \partial_\mu f$

$$S \to S + \frac{q}{c}\int \frac{\partial f}{\partial x^\mu}\, dx^\mu = S + \frac{q}{c} f \Big|_{\text{endpts.}}$$

$\Rightarrow$ Eqns of motion unchanged ✓

& $\vec{E}$ & $\vec{B}$ unchanged

The transformation of $\vec{E}$ & $\vec{B}$ can be written more generally as

$$\vec{E}_{\parallel} = \vec{E}'_{\parallel} \qquad \vec{B}_{\parallel} = \vec{B}'_{\parallel}$$

$$\vec{E}_{\perp} = \left(\vec{E}' - \frac{\vec{v}}{c} \times \vec{B}'\right)_{\perp} \gamma$$

$$\vec{B}_{\perp} = \left(\vec{B}' + \frac{\vec{v}}{c} \times \vec{E}'\right)_{\perp} \gamma$$

with $\parallel$ the compt of vector parallel to relative velocity of two moving frames and $\perp$ the compt of vector perpendicular to the relative motion.

Now trasf. back to observer frame $A^\mu = \Lambda^\mu{}_\nu A'^\nu$

$$\begin{pmatrix} \phi \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi' \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{aligned} \phi &= \gamma\phi' \\ A_x &= \beta\gamma\phi' \\ A_y &= 0 \\ A_z &= 0 \end{aligned}$$

Can write more gen'ly as $\vec{A} = \frac{\vec{v}}{c}\phi$

$$\phi = \frac{\gamma q}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}$$

Now compute $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla}\times\vec{A}$

$\rightarrow$ equiv to $E_x = E_x'$, $B_x = B_x'$,

$\vec{E}_\parallel = \vec{E}_\parallel'$, $\vec{B}_\parallel = \vec{B}_\parallel'$, $\vec{E}_\perp = \gamma\left(\vec{E}' - \frac{\vec{v}}{c}\times\vec{B}'\right)_\perp$

$\vec{B}_\perp = \gamma\left(\vec{B}' + \frac{\vec{v}}{c}\times\vec{E}'\right)_\perp$

Plug in $\vec{E}' = \frac{q}{r'^3}\left(\gamma x - \beta\gamma ct, y, z\right)$, $\vec{B}' = 0$

$\therefore \; E_x = q(\gamma x - \beta\gamma ct)/r'^3$

$E_y = \gamma q y / r'^3 \qquad E_z = \gamma q z / r'^3$

$\vec{B} = \frac{\vec{v}}{c}\times\vec{E}$

Define $\vec{r} \equiv \vec{X}_{obs} - \vec{X}_{part} = (x - vt, y, z)$

in observer frame

note $\vec{E} = \dfrac{\gamma q \vec{r}}{r'^3}$ ← radial from particle to observer

$$r'^2 = \gamma^2 (x-vt)^2 + y^2 + z^2 = \gamma^2 \left((x-vt)^2 + (1-\beta^2)(y^2+z^2) \right)$$

$$r^2 = (x-vt)^2 + y^2 + z^2$$

so $r'^2 = \gamma^2 r^2 \left(1 - \dfrac{\beta^2 (y^2+z^2)}{r^2} \right)$

Define $\vec{r} \cdot \vec{v} \equiv |r||v| \cos\psi$ ← angle between $\vec{r}$ & $\vec{v}$

then $\dfrac{y^2+z^2}{r^2} = \sin^2\psi$

So $r'^2 = \gamma^2 r^2 (1 - \beta^2 \sin^2\psi)$

$$\therefore \quad \boxed{\vec{E} = \left(\frac{q\vec{r}}{r^3} \right) \left(\frac{1 - v^2/c^2}{\left(1 - v^2/c^2 \sin^2\psi\right)^{3/2}} \right)}$$

↑ usual non-rel part ↑ relativistic correction

$$\boxed{\vec{B} = \frac{\vec{v}}{c} \times \vec{E}}$$

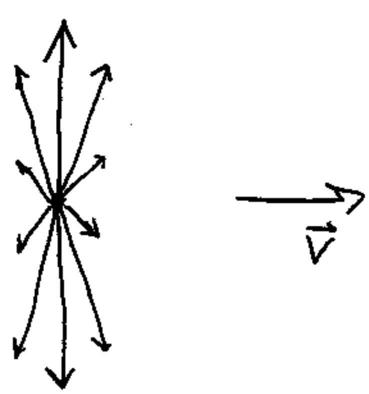
relativistic corr. factor: $\dfrac{1 - v^2/c^2}{\left(1 - v^2/c^2 \sin^2\psi\right)^{3/2}}$

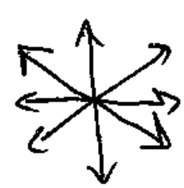
≈ 1 for $v \ll c$.

For $v \approx c$, very sensitive to $\sin^2\psi$.

For $\sin^2\psi \ll 1$, above factor $\rightarrow 1 - v^2/c^2 \sim 1/\gamma^2$ becomes very small.

For $\sin^2\psi \sim 1$, above factor $\rightarrow 1/\sqrt{1-v^2/c^2} = \gamma$ becomes very large. Picture field lines of moving charge. Always radial but magnitude greatly reduced, except in pancake $\perp$ to velocity vector. You can think of

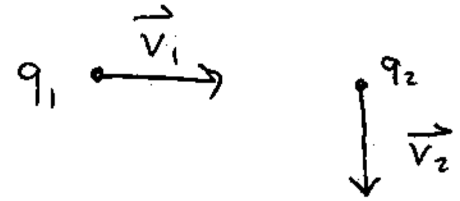


above drawing as usual  picture of field lines, but with length contraction of picture upon boosting.

Note $\vec{B} = \frac{\vec{v}}{c} \times \vec{E} \longrightarrow \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3}$ in non-rel limit

"Ampere-Biot-Savart" law for moving charge ✓

Fun to think about, e.g. in non-rel limit



$\vec{r}$ = vector pointing from 1 to 2

$\vec{V}_1 \parallel \vec{r}$, $\vec{V}_2 \perp \vec{r}$

Field due to q_1 @ q_2 location: $\vec{E}_1 = \dfrac{q_1 \vec{r}}{r^3}$

$\vec{B}_1 = \frac{1}{c}\vec{V}_1 \times \vec{E}_1 = 0$ since $\vec{V}_1 \parallel \vec{r}$

Field due to q_2 @ q_1 location $\vec{E}_2 = \dfrac{q_2(-\vec{r})}{r^3}$

$\vec{B}_2 = \dfrac{\vec{V}_2}{c} \times \vec{E}_2 \neq 0$ since $\vec{V}_2 \perp \vec{r}$

so force on 2 is $\vec{F}_2 = q_2 \vec{E}_1 + \frac{q_2}{c}\vec{V}_2 \times \vec{B}_1$

$\Rightarrow \vec{F}_2 = \dfrac{q_1 q_2 \vec{r}}{r^3}$

Force on 1 is $\vec{F}_1 = q_1 \vec{E}_2 + \frac{q_1}{c}\vec{V}_1 \times \vec{B}_2$

$$\vec{F}_1 = -\frac{q_1 q_2 \vec{r}}{r^3} - \frac{q_1 q_2}{r^3 c^2}\vec{V}_1 \times (\vec{V}_2 \times \vec{r})$$

extra sideways force, pointing in $(-\vec{V}_2)$ direction.

opposite to force on 2

Seems action $\neq$ reaction!

resolution: field energy/momentum.

$\mathcal{L}$ Example: $\vec{A} = B_0 x \hat{y}$, $\phi = 0$

$\hookrightarrow \vec{E} = 0 \qquad \vec{B} = B_0 \hat{z}$

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{q}{c}\vec{A}$$

$$\mathcal{H} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$ $\leftarrow$ const. since $\vec{A}$ & ϕ t indep.

$$\frac{d\vec{p}}{dt} = \nabla L = \frac{q}{c} V_y B_0 \hat{x}$$

$$\frac{d}{dt}(\gamma m V_x) = \frac{q}{c} V_y B_0 \qquad \gamma \equiv 1/\sqrt{1-v^2/c^2}$$

$$\frac{d}{dt}\left(\gamma m V_y + \frac{q}{c} B_0 x\right) = 0$$

$$\frac{d}{dt}(\gamma m V_z) = 0$$

$H = \text{const} \rightarrow \frac{d\gamma}{dt} = 0$
$\hookrightarrow |\vec{V}| = \text{const}$

$\hookrightarrow \frac{dV_x}{dt} = \Omega_c V_y \qquad \frac{dV_y}{dt} = -\Omega_c V_x \qquad V_z = \text{const}$

$\Omega_c \equiv qB/\gamma mc$ $\leftarrow$ cyclotron freq.

$\odot B$