# 4-vectors

e.g. $X^\mu \equiv (ct, \vec{x})$, $X_\mu \equiv (ct, -\vec{x})$

Lorentz inv't $(ct)^2 - \vec{x}\cdot\vec{x} \equiv X^\mu X_\mu$ — sum repeated upper & lower indices.

Consider more general 4-vectors $\begin{cases} a^\mu = (a_0, \vec{a}) \\ b^\mu = (b_0, \vec{b}) \end{cases}$
with $a^\mu = \Lambda^\mu_{\ \nu'} a^{\nu'}$

e.g. (like $X^\mu$) $\begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^{0'} \\ a^{1'} \\ a^{2'} \\ a^{3'} \end{pmatrix}$ & likewise for $b^\mu$

$\longrightarrow$ $a^\mu b_\mu \equiv (a_0 b_0 - \vec{a}\cdot\vec{b})$ is Lorentz inv't.

$a_\mu \equiv (a_0, -\vec{a})$ $\qquad b_\mu \equiv (b_0, -\vec{b})$

Raise & lower indices via metric $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$a_\mu = g_{\mu\nu} a^\nu \qquad a^\mu = g^{\mu\nu} a_\nu$

write $a\cdot b \equiv a^\mu b_\mu \equiv g_{\mu\nu} a^\mu b^\nu$. Lorentz inv't since $g_{\mu\nu} a^\mu b^\nu \longrightarrow (\Lambda^\mu_{\ \lambda'} g_{\mu\nu} \Lambda^\nu_{\ \sigma'}) a^{\lambda'} b^{\sigma'}$

but $\Lambda^\mu_{\ \lambda'} g_{\mu\nu} \Lambda^\nu_{\ \sigma'} = g_{\lambda'\sigma'}$

i.e. $\left(\begin{array}{cc|c} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & \\ \hline 0 & & \mathbb{1} \end{array}\right) \left(\begin{array}{cc|c} 1 & & 0 \\ & -1 & \\ \hline 0 & & -\mathbb{1} \end{array}\right) \left(\begin{array}{cc|c} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & \\ \hline 0 & & \mathbb{1} \end{array}\right) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ ✓

$\Longrightarrow$ $a\cdot b = a'\cdot b'$ is Lorentz inv't.

I.e. if a 4-vector $a^\mu \equiv (a_0, \vec{a})$ transforms as $a^\mu = \Lambda^\mu_{\ \nu'} a'^{\nu'}$, then $a_\mu \equiv (a_0, -\vec{a})$ transforms inversely, $a_\mu = (\Lambda^{-1})_\mu^{\ \nu'} a'^{\nu'}$

e.g. $\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\Lambda^{-1} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

s.t. $a^\mu b_\mu = a'^{\mu'} b'_{\mu'}$

Note that $x^\mu = \Lambda^\mu_{\ \nu'} x'^{\nu'} \Rightarrow \frac{\partial}{\partial x^\mu} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

transforms as $\partial_\mu$ w/ lower index, since

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^{\nu'}}{\partial x^\mu} \frac{\partial}{\partial x'^{\nu'}} = (\Lambda^{-1})_\mu^{\ \nu'} \frac{\partial}{\partial x'^{\nu'}}$$

$\Rightarrow \partial_\mu = (\Lambda^{-1})_\mu^{\ \nu'} \partial_{\nu'}$. Can raise via $g^{\mu\nu}$:

$$\partial^\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = \frac{\partial}{\partial x_\mu}.$$

Energy - momentum 4-vector $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ transforms w/ upper index (like $x^\mu$).

$\Rightarrow$ Can form Lorentz invts $p^\mu x_\mu = Et - \vec{p}\cdot\vec{x}$

or $p^\mu p_\mu = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2 c^2$

For massive particle $p^\mu = m u^\mu$

$u^\mu \equiv c \dfrac{dx^\mu}{ds}$ 4-velocity, transforms as 4-vector, like $dx^\mu$, since $c$ & $ds$ Lorentz invt.

$\boxed{\dfrac{ds}{c} \equiv d\tau \quad \text{proper time}}$

$ds = c\,dt\sqrt{1-v^2/c^2} \equiv c\,dt/\gamma \qquad \gamma \equiv \dfrac{1}{\sqrt{1-v^2/c^2}}$

$\hookrightarrow u^\mu = \gamma\left(c, \dfrac{d\vec{x}}{dt}\right) = \gamma(c, \vec{v})$

$p^\mu = m\gamma\left(c, \dfrac{d\vec{x}}{dt}\right) \equiv \left(\dfrac{E}{c}, \vec{p}\right)$

$\hookrightarrow E = mc^2\gamma \qquad \vec{p} = \gamma m\vec{v}$

$p^\mu p_\mu = \dfrac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2\gamma^2\left(1-\dfrac{\vec{v}^2}{c^2}\right) = m^2c^2$ ✓

Lorentz invt.

Force eqn: $\dfrac{dp^\mu}{d\tau} = f^\mu$

$\dfrac{d}{d\tau} = c\dfrac{d}{ds} = \gamma\dfrac{d}{dt}$

force 4-vector

so $\gamma \dfrac{d}{dt} p^\mu = f^\mu$

Fourier transf. / Q.M. plane waves $\sim e^{\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

$$= e^{-\frac{i}{\hbar} p_\mu x^\mu}$$

$$\therefore \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \longrightarrow -\frac{i}{\hbar} p_\mu$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad p_\mu = \left(\frac{E}{c}, -\vec{p}\right)$$

$$\text{so} \quad E \Rightarrow i\hbar \frac{\partial}{\partial t} \qquad \vec{p} \rightarrow -i\hbar\vec{\nabla}$$

as is familiar from Q.M.

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Besides $a\cdot b$, can form Lorentz invts via

$\varepsilon^{\mu\nu\lambda\sigma}$ totally antisymm w/ $\varepsilon^{0123} = 1$

Under Lorentz boost, $\varepsilon^{\mu\nu\lambda\sigma} \rightarrow \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \Lambda^\lambda_{\lambda'} \Lambda^\sigma_{\sigma'} \varepsilon^{\mu'\nu'\lambda'\sigma'}$

$$= (\det\Lambda)\, \varepsilon^{\mu'\nu'\lambda'\sigma'} = \varepsilon^{\mu'\nu'\lambda'\sigma'}$$

since $\det\Lambda = 1$ (e.g. $\Lambda = \left(\begin{array}{cc|c} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{1} \end{array}\right)$

$\therefore$ $a_\mu b_\nu c_\lambda d_\sigma \varepsilon^{\mu\nu\lambda\sigma}$ is Lorentz invt

for any 4 vectors $a_\mu, b_\nu, c_\lambda, d_\sigma$

# Relativistic Mechanics.

$S \equiv \int dt\, L$ must be Lorentz inv't.

Simplest possibility $\quad S = \alpha \int ds$

$$ds = \sqrt{c^2 dt^2 - d\vec{x}^2} = c\,dt\sqrt{1-\frac{v^2}{c^2}}$$

gives $\quad L = \alpha c\sqrt{1-\frac{v^2}{c^2}} \quad \approx \alpha c - \frac{1}{2}\frac{\alpha}{c}v^2 + \dots$

for $v \ll c$

expect $\quad L \approx \frac{1}{2}mv^2 - mc^2 \quad \Rightarrow \quad \alpha = -mc$

so the relativistic action of a free particle of mass $m$ is $\boxed{S = -mc\int ds}$

$\boxed{\text{i.e. } L = -mc^2\sqrt{1-\frac{v^2}{c^2}}}$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

$$H = \vec{p}\cdot\vec{v} - L = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$ $\leftarrow$ (only true for $m \neq 0$!)

note $\quad \vec{p} = H\frac{\vec{v}}{c^2} \quad \leftarrow$ true even for $m=0$

Can write $\quad p_\mu = -\partial S/\partial x^\mu$

Now couple charged particle to E & M

via $A^\mu \equiv (\phi, \vec{A})$ = 4-potential (transforms as $x^\mu$)

$$S = \int \left(-mc\,ds - \frac{q}{c} A_\mu dx^\mu\right) \quad \text{Lorentz inv.}$$

$$\Rightarrow S \equiv \int dt\, L \quad \text{with} \quad L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$$

~~Since~~ $\Rightarrow \vec{p} = \dfrac{\partial L}{\partial \vec{v}} = \dfrac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \dfrac{q}{c}\vec{A}$

$$\mathcal{H} = \vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L = \frac{mc^2}{\sqrt{1-v^2/c^2}} + q\phi$$

$$\hookrightarrow \mathcal{H} = c\sqrt{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + m^2c^2} + q\phi$$

Eqns of motion: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \vec{v}}\right) = \dfrac{\partial L}{\partial \vec{x}}$

$$\hookrightarrow \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{q}{c}\vec{A}\right) = \frac{q}{c}\frac{\partial(\vec{A}\cdot\vec{v})}{\partial \vec{x}} - q\nabla\phi$$

Write $\dfrac{d\vec{A}}{dt} = \dfrac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$

$$\rightarrow \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$$

with $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla}\times\vec{A}$

(using $\nabla(\vec{A}\cdot\vec{v}) - (\vec{v}\cdot\vec{\nabla})\vec{A} = \vec{v}\times(\vec{\nabla}\times\vec{A})$ )

$\hookrightarrow$ Lorentz force law ✓

Write as 4-vector eqns for $p^\mu = m u^\mu$

$$\frac{d}{d\tau}p^\mu = f^\mu$$

$p^\mu \equiv (p_0, \vec{p})$

$u^\mu \equiv (u_0, \vec{u}) \equiv \gamma(c, \vec{v})$

$$\frac{d}{d\tau} = \gamma\frac{d}{dt}$$

so $\frac{d}{d\tau}\vec{p} = \frac{q}{c}\left(u_0\vec{E} + \vec{u}\times\vec{B}\right)$

Lorentz force eqn

also get $\frac{d}{d\tau}p_0 = \frac{q}{c}\vec{u}\cdot\vec{E}$ energy change

only Electric field does work!

together form $\frac{d}{d\tau}p^\mu = f^\mu$

w/ force $f^\mu = \frac{q}{c}\left(\vec{u}\cdot\vec{E},\ u_0\vec{E} + \vec{u}\times\vec{B}\right)$

* Exercise: suppose $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$ $\quad y = y'$, $z = z'$

1) Using the fact that $A^\mu \equiv (\phi, \vec{A})$ transforms as a 4-vector (like $x^\mu = (ct, \vec{x})$) and the expressions for $\vec{E}$ & $\vec{B}$ in terms of $\phi$ & $\vec{A}$,

$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$, show for the above transf. that

a) $E_x = E_x'$ $\quad \leftarrow \left( \text{i.e. } -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = -\frac{\partial \phi'}{\partial x'} - \frac{1}{c}\frac{\partial A_x'}{\partial t'} \right.$

b) $B_x = B_x'$

c) $$\begin{pmatrix} E_y \\ B_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_y' \\ B_z' \end{pmatrix}$$

d) $$\begin{pmatrix} E_z \\ B_y \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_z' \\ B_y' \end{pmatrix}$$

2) Verify that $f^\mu = \frac{q}{c}\left(\vec{u}\cdot\vec{E},\ u_0\vec{E} + \vec{u}\times\vec{B}\right)$ transforms as a 4-vector, i.e. for the above transf.

$$\begin{pmatrix} f^0 \\ f^1 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} f^{0\prime} \\ f^{1\prime} \end{pmatrix} \qquad f^2 = f'^2, \quad f^3 = f'^3$$