Can get directly from $\nabla^2 G = -4\pi\,\delta(\vec{x}-\vec{x}')$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\,\delta(r-r')\,\delta(\Omega-\Omega')$$

$$= \frac{1}{r^2}\,\delta(r-r') \sum_{l,m} Y_{lm}^*(\Omega')\,Y_{lm}(\Omega)$$

$$G(\vec{x},\vec{x}') = \sum_{l,m} g_l(r,r')\,Y_{lm}^*(\Omega')\,Y_{lm}(\Omega)$$

$$\nabla^2 * \longrightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r*) - \frac{1}{r^2}\hat{L}^2 *$$

so $$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r g_l) - \frac{l(l+1)}{r^2} g_l = -\frac{4\pi}{r^2}\delta(r-r')$$

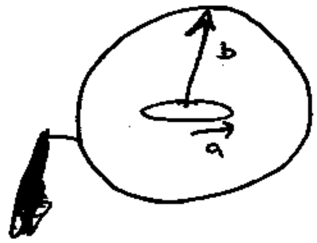
$$g_l(r,r') = \begin{cases} A_l r^l + B_l r^{-(l+1)} & r<r' \\ A_l' r^l + B_l' r^{-(l+1)} & r>r' \end{cases}$$

(A, B, A', B' dep. on r')

get general soln for shell bounded by $r=a$, $r=b$ $\quad a<b$

$$g_l(r,r') = \frac{4\pi}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

e.g.



circular ring of charge inside conducting sphere @ $V(\theta,\phi)$

$$\rho(\vec{x}') = \frac{Q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \underset{\underset{V(\theta',\phi')}{\uparrow}}{\Phi(\vec{x}')}\nabla G\cdot d\vec{S}$$

Use $$G(\vec{x},\vec{x}') = \sum_{\ell,m}\frac{4\pi}{2\ell+1}\,r_<^{\ell}\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)Y_{\ell m}^*(\Omega')\,Y_{\ell m}(\Omega)$$

azimuthal symm $\Rightarrow$ only $m=0$ term contributes after above integrals. $Y_{\ell,0} = \sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)$

so replace $$G(\vec{x},\vec{x}') \longrightarrow \sum_{\ell=0}^{\infty} r_<^{\ell}\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)P_\ell(\cos\theta')\cdot P_\ell(\cos\theta)$$

$$\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' = \frac{Q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty} r_<^{\ell}\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)P_\ell(0)\,P_\ell(\cos\theta)$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\,r_<^{2n}\left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

Using $$P_\ell(0) = \begin{cases} 0 & \ell \text{ odd} \\ (-1)^n\binom{2n}{n} & \ell = 2n \end{cases}$$

Also $\Phi = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\int_S \Phi(\vec{x}')\,\nabla G\,d\vec{s}$

just evaluated evaluate now

$$\left.\frac{\partial G}{\partial r'}\right|_{r=b} = -\frac{4\pi}{b^2}\sum_{l,m}\left(\frac{r}{b}\right)^l Y^*_{lm}(\Omega')\,Y_{lm}(\Omega)$$

$$d\vec{S} \rightarrow \hat{n}\,r^2\,d\Omega'$$

so $-\frac{1}{4\pi}\int \Phi(\vec{x}')\,\nabla G\cdot d\vec{S} = \sum_{l,m}\left[\int V(\Omega')\,Y^*_{lm}(\Omega')\,d\Omega'\right]$

$\cdot\left(\frac{r}{b}\right)^l Y_{lm}(\Omega).$

Suppose Grounded $\rightarrow V(\Omega') = 0$

$$\Phi = \frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n} r_<^{2n}\left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

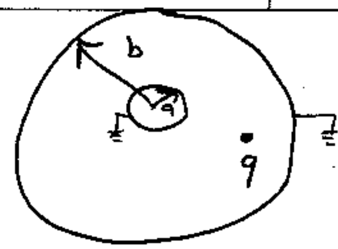
Find surface charge σ on sphere.

$$\sigma = -\varepsilon_0\nabla\phi\cdot\hat{n}\Big|_{r_>=b} = \varepsilon_0\left.\frac{\partial\phi}{\partial r}\right|_{r_>=b}$$

$$= -\frac{Q}{4\pi}\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}a^{2n}\left(\frac{2n+1}{b^{2n+2}} + \frac{2n}{b^{2n+2}}\right)P_{2n}(\cos\theta)$$

so $\int \sigma\, b^2 d(\cos\theta)\,d\phi = 2\pi b^2\int_{-1}^{1} d(\cos\theta)\,\sigma = -Q$ ✓

Another example:
Charge q between grounded spheres.



Find surface charges induced on inner & outer spheres. Let charge q be at position $\vec{x}'$

potential $\Phi(\vec{x}) = \dfrac{q}{4\pi\varepsilon_0} G(\vec{x},\vec{x}') \leftarrow$ our Green fn. from last time

$$\Phi(\vec{x}) = \frac{q}{\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \cdot$$

$$\cdot \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right) Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

Inner sphere $\sigma_{in} = -\varepsilon_0 \nabla\Phi\cdot\hat{n} = -\varepsilon_0 \dfrac{\partial \Phi}{\partial r_<}\Big|_{\substack{r_< = a \\ r_> = r'}}$

$$= -q \sum_{\ell,m} \frac{1}{(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \left(\ell a^{\ell-1} + (\ell+1) a^{\ell-1}\right)$$

$$\left(\frac{1}{r'^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right) Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega)$$

$$= -\frac{q}{4\pi} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)a^{\ell-1}}{\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \left(\frac{1}{r'^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right) P_\ell(\cos\gamma)$$

Total charge on inner sphere

$$Q_{in} = a^2 \int \sigma \, d\Omega = -q \sum_{\ell m} \frac{a^{\ell+1}}{\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)}$$

$$\left(\frac{1}{r'^{\ell+1}} - \frac{r'^{\ell}}{b^{2\ell+1}}\right) Y^*_{\ell m}(\Omega') \int d\Omega \, Y_{\ell m}(\Omega)$$

Use $\int d\Omega \, Y^*_{\ell' m'}(\Omega) Y_{\ell m}(\Omega) = \delta_{\ell\ell'} \delta_{mm'}$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

So $\int d\Omega \, Y_{\ell m}(\Omega) = \sqrt{4\pi} \int d\Omega \, Y_{\ell m}(\Omega) Y_{00}(\Omega)$

$= \sqrt{4\pi} \, \delta_{\ell,0} \, \delta_{m,0}$

So $\boxed{Q_{in} = -q \frac{a}{\left(1-\left(\frac{a}{b}\right)\right)} \left(\frac{1}{r'} - \frac{1}{b}\right)}$

note for $b \to \infty$ $\quad Q_{in} \to -\frac{qa}{r'} = q_{image}$

from earlier lecture

Likewise $\sigma_{out} = -\varepsilon_0 \nabla\Phi \cdot \hat{n} = \varepsilon_0 \left.\frac{\partial \Phi}{\partial r_>}\right|_{\substack{r_> = b \\ r_< = r'}}$

$$= -q \sum_{\ell m} \frac{1}{(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \left(r'^{\ell} - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \cdot$$

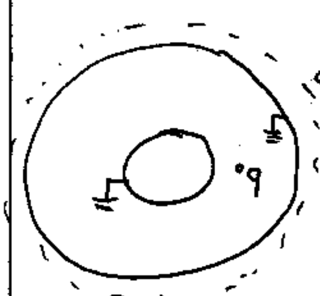
$$\left(\frac{(2\ell+1)}{b^{\ell+2}}\right) Y^*_{\ell m}(\Omega') \, Y_{\ell m}(\Omega)$$

$$Q_{out} = b^2 \int d\Omega \, \sigma_{out} = -q \left(\frac{1}{1-\left(\frac{a}{b}\right)}\right)\left(1 - \frac{a}{r'}\right)$$

note for $a \to 0$ $\quad Q_{out} \to -q$

$$Q_{in} + Q_{out} = -\frac{q}{1-\frac{a}{b}}\left(\frac{a}{r'} - \frac{a}{b} + 1 - \frac{a}{r'}\right)$$

$= -q.$ This was to be expected:



$\leftarrow$ put whole sys in Gauss surface

$\Phi \equiv 0$ on this surface so

Gauss' law $\Rightarrow$ $Q_{inside} = Q_{in} + Q_{out} + q$

$= 0$

$\Rightarrow \quad Q_{in} + Q_{out} = -q \ \checkmark$

Now consider cylindrical coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Solve $\nabla^2 \Phi = 0$ via $\Phi = R(\rho) Q(\phi) Z(z)$

$$\frac{d^2 Z}{dz^2} = k^2 Z \qquad \frac{d^2 Q}{d\phi^2} = -\nu^2 Q$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0$$

$$Z(z) = e^{\pm kz} \qquad Q(\rho) = e^{\pm i\nu\phi}$$

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \qquad (x \equiv k\rho)$$

Bessel eqn, sol'ns are Bessel fns

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \qquad \text{eg } \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ix\sin\theta} = J_0(x)$$

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad \text{Neumann fn.}$$

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + iN_\nu(x) \qquad H_\nu^{(2)} = J_\nu - iN_\nu \qquad \text{Hankel fn.}$$

All J_ν, N_ν, $H_\nu^{(1)}$, $H_\nu^{(2)}$ satisfy

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\,\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\,\frac{d\Omega_\nu(x)}{dx}$$

e.g. $J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$

$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x$

for $x \ll 1$ $\quad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \qquad N_{\nu\neq 0}(x) \to -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu$

$N_0(x) \to \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + .577\right]$

Each $J_\nu(x)$ has ∞ # of roots

$J_\nu(x_{\nu n}) = 0 \qquad n = 1, 2, 3, \ldots$

e.g. $x_{0,n} = 2.405,\ 5.520,\ 8.654, \ldots$

for n large $\quad x_{\nu n} \simeq \pi n + \left(\nu - \frac{1}{2}\right)\pi/2$

Completeness: $\quad f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\, \rho/a\right)$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^\infty d\rho\, \rho\, f(\rho)\, J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)$$

Cylindrical Green fn $\qquad \nabla^2 G = -4\pi\, \delta(\vec{x} - \vec{x}')$

$$= -\frac{4\pi}{\rho}\, \delta(\rho - \rho')\, \delta(\phi - \phi')\, \delta(z - z')$$

$$\delta(z - z') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(z-z')}, \qquad \delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \sum_m e^{im(\phi-\phi')} e^{ik(z-z')}$$

$g_m(k, \rho, \rho')$

w/ $\frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{d g_m}{d\rho}\right) - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho} \delta(\rho-\rho')$

$g(k, \rho, \rho') = \Psi_1(k\rho_<)\, \Psi_2(k\rho_>)$

$\Psi_{1,2}$ lin combo of I_m & K_m

$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$ imag. arg.
$I_\nu(x) = i^{-\nu} J_\nu(ix)$

Lot's of other special functions occur in solving $\nabla^2 \Phi = 0$ or $\nabla^2 \Phi = -4\pi \delta()$ in different spaces (boundary conditions)

hypergeometric functions etc...