Electrodynamics

Basic eqns: $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \end{array}\right] \ (1)$$

Lorentz 4-vector of eqns

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \end{array}\right] \ (2)$$

4 vector

☆ (1) $\Rightarrow$ electric charge conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \quad \Rightarrow \frac{\partial}{\partial t}\int_V \rho\, dV = -\int_{\partial V} \vec{J}\cdot d\vec{S}$$

☆ Compatible with relativity

☆ linear $\Rightarrow$ superposition!

Contrast with gravity: relativistic version = general relativity, Einstein's eqns: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$

horribly nonlinear eqn for fields, difficult to solve, since no superposition.

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Another force = strong nuclear force. Classical eqns similar to above Maxwell eqns, but 8 copies of $\vec{E}^a$ & $\vec{B}^a$ $a = 1 \cdots 8$ and also extra nonlinear terms, e.g. $\sim \vec{E}^a \times \vec{E}^b$.

$\Rightarrow$ No superposition, much harder to solve.

Each force is carried by a particle:

Force	Source	force carrier
E&M	electric charges	photon
gravity	energy	graviton
strong nuclear	color charge	gluons
weak nuclear	weak charge	$W^\pm$, Z^0 (roughly)

- photon = electrically neutral. Why linear & superposition works. Exp'l bounds $q_\gamma < 5 \times 10^{-30}\, q_e$

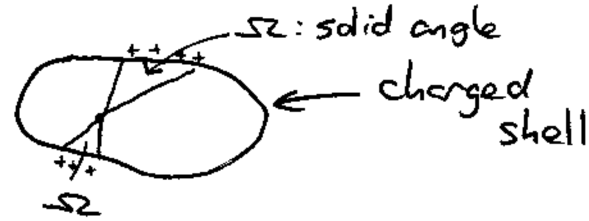
For all other known forces above, force carrier carries source charge, not neutral.
$\hookrightarrow$ nonlinearities

- photon = massless. Exp'l bound : $m_\gamma < 10^{-24} m_e$

  ↳ only 2 polarizations ✓

  ↳ why we can see distant stars

  ↳ $F \sim \frac{q_1 q_2}{r^2}$



no force at any place
in charged shell since charges on opposite sides
$q_2 \sim r^2 \Omega$ lead to balancing forces

$$F \sim \frac{q_1 \cancel{r^2}}{\cancel{r^2}} (\Omega - \Omega) = 0.$$

(First observed by Ben Franklin)

Works Only for massless photon!

The reason why superconductors have such unusual properties, e.g. persistent currents, Meissner effect of magnetic flux expulsion, etc. is that the photon is effectively massive in a superconductor. Shows how different life would be with a massive photon.

Maxwell's eqns ② $\rightarrow$ no magnetic monopoles yet observed. Many theories predict monopoles, e.g. GUT grand unification of E&M with nuclear forces.

$\hookrightarrow$ Many monopoles should have been produced in early Universe. Why don't we see any? Most would be swept outside of our visible horizon by inflation. So eventually ② should be perhaps modified to allow magnetic monopole sources on RHS. Easiest way to write ①+②

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot(\vec{E}+i\vec{B}) = 4\pi(\rho_e + i\rho_m) \\ -i&\vec{\nabla}\times(\vec{E}+i\vec{B}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}+i\vec{B}) = \frac{4\pi}{c}(\vec{J}_e + i\vec{J}_m) \end{aligned}}$$

note can rotate

$$\left.\begin{aligned} \vec{E}+i\vec{B} &\rightarrow e^{i\alpha}(\vec{E}+i\vec{B}) \\ \rho_e + i\rho_m &\rightarrow e^{i\alpha}(\rho_e + i\rho_m) \\ \vec{J}_e + i\vec{J}_m &\rightarrow e^{i\alpha}(\vec{J}_e + i\vec{J}_m) \end{aligned}\right\} \text{preserves for any phase } \alpha$$

e.g. $e^{i\alpha} = -i$:

$$\left[\begin{aligned} &\vec{E}\rightarrow\vec{B}, \quad \vec{B}\rightarrow -\vec{E} \\ &\rho_e \rightarrow \rho_m, \quad \rho_m \rightarrow -\rho_e \\ &\vec{J}_e \rightarrow \vec{J}_m, \quad \vec{J}_m \rightarrow -\vec{J}_e \end{aligned}\right.$$

"Electric-Magnetic duality"

E.g. point electric charge (at $\vec{r}=0$) has $\vec{E} = \frac{q_e \hat{r}}{r^2}$

$\therefore$ point magnetic charge @ $\vec{r}=0$ has $\vec{B} = \frac{q_m \hat{r}}{r^2}$.

No magnetic sources simplifies life, though.

If $\rho_m = \vec{J}_m = 0$, then ② are the familiar

2 Maxwell eqns

$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

Can solve these via

$$\vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \nabla \times \vec{A}$$

$(\phi, \frac{1}{c}\vec{A})$ transforms as a 4-vector.

☆ "Gauge invariance" $\vec{E}$ & $\vec{B}$ are unchanged by

$$\left.\begin{aligned} \vec{A} &\to \vec{A} + \nabla f \\ \phi &\to \phi - \frac{1}{c}\frac{\partial f}{\partial t} \end{aligned}\right] \text{ for } \underline{\text{any}} \text{ function } f(t, \vec{x}).$$

Gauge invariance is the key to modern understanding of all forces $\longleftrightarrow$ local gauge symmetries.

Can use gauge inv. to impose a convenient gauge, via choice of appropriate $f(\vec{x},t)$.

popular choices: "Coulomb gauge" $\vec{\nabla}\cdot\vec{A} \stackrel{!}{=} 0$

or "Lorentz gauge:" $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} \stackrel{!}{=} 0$

We'll soon see $\vec{A}$ ¦ ϕ enter the Lagrangian

as $L = L_0 + \frac{q}{c}\frac{d\vec{x}}{dt}\vec{A} - q\phi$

↑ potential energy

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \underset{m\vec{v}}{\underset{\parallel}{\vec{p}_0}} + \frac{q}{c}\vec{A}$$

$$H = \vec{p}\cdot\vec{v} - L = H_0 + q\phi$$

Under gauge transf. $\vec{A} \to \vec{A} + \nabla f$
$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}$

$$L \to L + \frac{q}{c}\left(\frac{d\vec{x}}{dt}\cdot\nabla f + \frac{\partial f}{\partial t}\right) = L + \frac{q}{c}\frac{df}{dt}$$

total deriv

$\therefore$ the action $S = \int dt\, L \to S + \frac{q}{c} f(\vec{x},t)$

$\to$ Equations of motion unchanged!

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But in Q.M. the wavefunction $\psi \sim e^{\frac{i}{\hbar}S}$

so $\psi \to e^{\frac{iq}{\hbar c}f(\vec{x},t)}\,\psi$ picks up a space/time dependent phase. Probabilities $\sim |\psi|^2$ are unchanged, phase unobservable.

E & M = "U(1) gauge theory." Turn around logic: declare symmetry $\psi \to e^{\frac{iq}{\hbar c}f(\vec{x},t)}\psi$

$$\vec{P}_0\psi = \frac{\hbar}{i}\nabla\psi \to \left[\frac{\hbar}{i}(\nabla\psi) + \frac{q}{c}(\nabla f)\psi\right]e^{\frac{iq}{\hbar c}f}$$

doesn't transform nicely. Introduce $\vec{A}$ with

$\vec{A} \to \vec{A} + \nabla f$ & replace $\vec{P}_0 \to \vec{P} = \vec{P}_0 + \frac{q}{c}\vec{A}$

$$\vec{P}_0\psi = \left(\vec{P}_0 - \frac{q}{c}\vec{A}\right)\psi \to e^{\frac{iq}{\hbar c}f}\left(\vec{P}_0 - \frac{q}{c}\vec{A}\right)\psi$$

$\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$ (e.g. $\vec{p} - \frac{q}{c}\vec{A} = m\vec{v}$) transforms nicely.

Likewise $H\psi = i\hbar\frac{\partial\psi}{\partial t}$ replace $i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} + q\phi$

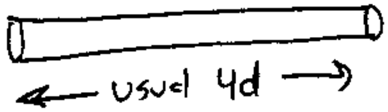
$q = nq_e$ charge quantization $\to f \sim f + \frac{2\pi\hbar c}{q_e}$

f naturally lives on a circle of radius

$R = \frac{\hbar c}{q_e}$ iff charge is quantized.

Kaluza Klein idea: this circle is an actual 5th

dimension



5th dim $R = (\hbar c / q_e)\, l_{Pl}$

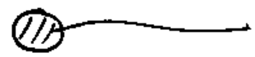
$\sim 10^{-33}$ cm

Would "explain" charge quantization.

Another possible explanation of charge quantization:

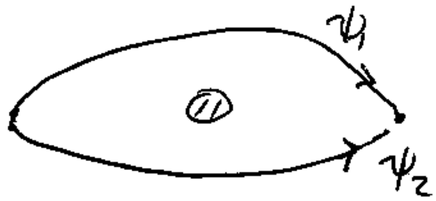
$\exists$ a magnetic monopole somewhere in the Universe.

Recall Aharonov Bohm



$B \neq 0$ inside solenoid.

$B = 0$ outside



$$\psi_1 = e^{\frac{iq}{\hbar c}\Phi}\, \psi_2$$

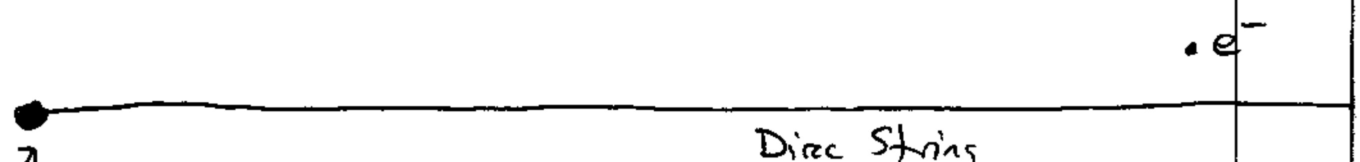
$$\Phi = \oint \vec{A}\cdot d\vec{\ell} = \int \nabla \times \vec{A} \cdot d\vec{S}$$

$$= \int \vec{B}\cdot d\vec{S} \quad \text{magnetic flux}$$

Use $\psi \sim e^{\frac{i}{\hbar}\int L\, dt}$

$L = L_0 + \frac{q}{c}\frac{d\vec{x}}{dt}\cdot \vec{A} - q\phi,$

Consider a magnetic monopole as the end of a fictitious solenoid = "Dirac String"



More distant e^- around string: $\psi \to e^{\frac{iq\Phi}{\hbar c}}\psi \stackrel{!}{=} \psi$

For ψ to be well defined, need

$$\frac{q_e \Phi}{\hbar c} \overset{!}{=} 2\pi n \qquad n = \text{integer}$$

Monopole with magnetic charge g_m has

$$\vec{B} = \frac{g_m \hat{r}}{r^2} \quad \& \quad \therefore \text{ flux } \Phi = \int \vec{B} \cdot d\vec{A}$$

$$\Phi = 4\pi g_m . \quad \therefore \quad \boxed{2 \frac{q_e g_m}{\hbar c} \overset{!}{=} \text{integer}}$$

"Dirac quantization cond"

$\therefore$ Smallest unit of magnetic charge $g_m = \frac{\hbar c}{2 q_{elec}}$

Working backwards: if $\exists$ anywhere in Universe a magnetic monopole w/ magnetic charge g_m, must have all electric charges quantized as $q = n\left(\frac{\hbar c}{2 g_m}\right)$. Take $\frac{\hbar c}{2 g_m} = q_{electron}$

Similar mech. in grand unified theories. Charge quantized (similar to Q.M. angular momentum quantization) & magnetic monopoles. Mass of monopoles ~ $\frac{1}{\alpha_{GUT}} M_{GUT}$. $\frac{1}{\alpha_{GUT}} \sim 25$ $\quad M_{GUT} \sim 10^{16}$ GeV

$\therefore M_{monopole} \sim 10^{18} M_{proton}$